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Lightlike hypersurfaces of a semi-Riemannian manifold with a semi-symmetric metric connection

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ABSTRACT

In this study, we show that the connection induced on a lightlike hypersurface of a semi-Riemannian manifold with a semi-symmetric metric connection is semi-symmetric but not a metric connection and obtain the equations of Gauss and Codazzi. Then, we study conditions under which the Ricci tensor is symmetric and parallel.

Keywords: Gauss and Codazzi equations, Ricci tensor, Levi-Civita connection, Semi-symmetric connection. Lightlike hypersurface,

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INTRODUCTION

Hayden (1932) introduced a semi-symmetric metric connection on a Riemannian manifold. Yano (1970) proved the theorem: *In order that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat*. Some topics related to this theorem were studied by Imai (1972_a). Imai (1972_b) gave basic properties of a hypersurface of a Riemannian manifold with a semi-symmetric metric connection and got conformal equations of Gauss and Codazzi.

Duggal and Sharma (1986) studied a semi-symmetric metric connection in a semi-Riemannian manifold. In this work, they showed that there exists an interplay between the Riemannian and semi-Riemannian geometry with respect to a semi-symmetric metric connection.

Konar and Biswas (2001) considered a semi-symmetric metric connection on Lorentz manifolds. They showed that the perfect fluid space time with a nonzero constant scalar curvature which admits a semi-symmetric metric connection whose Ricci tensor is zero and has vanishing expansion scalar and acceleration vector.

It is well known that lightlike hypersurfaces are of metrics with vanishing determinants and this degeneracy of metrics leads to several difficulties. Firstly, the contravariant metric cannot immediately be defined, so the connection cannot be specified uniquely in the normal way. Secondly, the normal is a lightlike vector lying in the tangent plane, which makes it necessary to look for some other vector to rig the hypersurface, and makes it impossible to normalize the normal in the usual way. Since these objects are considered, study of the general theory of lightlike hypersurfaces is very important. Several papers have been written on lightlike hypersurfaces in recent years (Katsuno 1980, Bejancu 1996, Duggal & Bejancu 1996, Günes et al. 2003).

In the present paper, we study lightlike hypersurfaces of a semi-Riemannian manifold with a semi-symmetric metric connection. We have proved that on lightlike hypersurfaces the connection induced from the semi-symmetric metric connection is semi-symmetric but not a metric connection, and also on the screen distribution the connection induced from that connection is a metric connection. For a lightlike hypersurface and screen distribution we define the induced geometrical objects with respect to a semi-symmetric connection such as second fundamental form, shape operator, etc. Then we give the equations of Gauss and Codazzi. Finally, we have a necessary and sufficient condition for the Ricci tensor of a lightlike hypersurface with respect to a semi-symmetric connection to be symmetric, and we also give some conditions for the Ricci tensor of a lightlike hypersurface with respect to a semi-symmetric connection is symmetric and parallel in a semi-Riemannian space form case.

PRELIMINARIES

Let M be a hypersurface of an $(n+1)$-dimensional, $n > 1$, semi-Riemannian manifold $\tilde{M}$ with semi-Riemannian metric $\tilde{g}$ of index $1 \leq \nu \leq n$. We consider:

$$T_x^{\perp} M = \{ Y_x \in T_x \tilde{M} \,|\, \tilde{g}_x(Y_x, X_x) = 0, \, \forall X_x \in T_x M \}$$

for any $x \in M$. Then we say that M is a *lightlike (null, degenerate) hypersurface of* $\tilde{M}$ or equivalently, the immersion

$$i : M \to \tilde{M}$$

of M in $\tilde{M}$ is *lightlike (null, degenerate)* if $T_x M \cap T_x M^{\perp} \neq \{0\}$ at any point $x \in M$. Henceforth we identify $i(M)$ with M and we denote the differential di, immersing a vector field X on M to a vector field ϕX on $\tilde{M}$ by ϕ. Thus the induced metric tensor $g = \tilde{g}|_M$ is defined by:

$$g(X, Y) = \tilde{g}(\phi X, \phi Y), \ \forall X, Y \in \Gamma(TM).$$

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is non-degenerate subbundle of TM which is called the screen distribution on M and denoted by $S(TM)$. We have the following splitting into an orthogonal direct sum:

$$TM = S(TM) \perp TM^{\perp}. \tag{1}$$

The subbundle $S(TM)$ is non-degenerate, so is $S(TM)^{\perp}$, and the following holds:

$$T\tilde{M} = S(TM) \perp S(TM)^{\perp}, \tag{2}$$

where $S(TM)^{\perp}$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\tilde{M}|_M$.

Let $tr(TM)$ denote a complementary vector bundle of $TM^{\perp}$ in $S(TM)^{\perp}$. Then we have:

$$S(TM)^{\perp} = TM^{\perp} \oplus tr(TM). \tag{3}$$

Let U be a coordinate neighborhood in M and ξ be a basis of $\Gamma(TM^{\perp}|_U)$. Then there exists a basis N of $tr(TM)|_U$ satisfying the following conditions:

$$\tilde{g}(N, \xi) = 1, \text{and } \tilde{g}(N, N) = \tilde{g}(W, N) = 0, \forall W \in \Gamma(S(TM)|_U).$$

The subbundle $tr(TM)$ is called a *lightlike transversal vector bundle* of M. We note that $tr(TM)$ is never orthogonal to TM. From (1), (2) and (3) we have:

$$T\tilde{M}|_M = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM) \tag{4}$$

(Duggal & Bejancu 1996).

Example 2.1 (Duggal & Bejancu 1996). *Consider the unit pseudoshpere* S_1^3 of R_1^4 *given by the equation:*

$$-(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1.$$

Cut S_1^3 *by the hyperplane* $x_1 - x_2 = 0$, *and obtain a lightlike surface* M of S_1^3 *with:*

$$TM^{\perp} = Span\{\xi = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\}.$$

Then take as the screen distribution:

$$S(TM) = \text{Span}\{\text{W}\},$$

where $W = x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}$. *It follows that tr(TM) corresponding to S(TM) is spanned by:*

$$N = -\frac{1}{2}\{(1+(x_1)^2\frac{\partial}{\partial x_1} + ((x_1)^2-1)\frac{\partial}{\partial x_2} + 2x_1x_3\frac{\partial}{\partial x_3} + 2x_1x_4\frac{\partial}{\partial x_4}\}.$$

SEMI-SYMMETRIC METRIC CONNECTION

For $n > 1$, let $\tilde{M}$ be an $(n+1)$-dimensional differentiable manifold of class C^{∞} and $\tilde{\nabla}$ a linear connection in $\tilde{M}$. The torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is given by:

$$\tilde{T}(\tilde{X},\tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X} - [\tilde{X},\tilde{Y}]\ ,\ \ \forall\tilde{X},\tilde{Y}\in\Gamma\big(T\tilde{M}\big)$$

and have type (1,2). When the torsion tensor $\tilde{T}$ satisfies:

$$\tilde{T}(\tilde{X},\tilde{Y}) = \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{\pi}(\tilde{X})\tilde{Y}$$

for a 1 -form $\tilde{\pi}$, the connection $\tilde{\nabla}$ is said to be *semi-symmetric* (Yano 1970).

Let us consider a semi-Riemannian metric $\tilde{g}$ of index ν with $1 \le \nu \le n$ in $\tilde{M}$ and $\tilde{\nabla}$ satisfying:

$$\tilde{\nabla}\tilde{g} = 0.$$

A linear connection of this type is called a *metric connection* (O'Neill. 1983).

Now suppose that the semi-Riemannian manifold $\tilde{M}$ admits a semi-symmetric metric connection which is given by:

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\pi}\big(\tilde{Y}\big)\tilde{X} - \tilde{g}\big(\tilde{X},\tilde{Y}\big)\tilde{Q} \tag{5}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ on $\tilde{M}$, where $\overset{\circ}{\tilde{\nabla}}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric $\tilde{g}, \tilde{\pi}$ is a $1-$ form and $\tilde{Q}$ is the vector field defined by:

$$\tilde{g}(\tilde{Q},\tilde{X}) = \tilde{\pi}(\tilde{X})$$

for an arbitrary vector field $\tilde{X}$ on $\tilde{M}$ (Yano 1970, Duggal & Sharma 1986). Using (4), we can write:

$$\tilde{Q} = \phi Q + \mu N \tag{6}$$

where Q is a vector field and μ is a function on M.

Let us denote the symmetric linear connection induced on the lightlike hypersurface from the Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$ by $\overset{\circ}{\nabla}$. Then we have:

$$\overset{\circ}{\tilde{\nabla}}_{\phi X} \phi Y = \phi (\overset{\circ}{\nabla}_X Y) + B(X, Y) N \tag{7}$$

for arbitrary vector fields X and Y on M, where B is the local second fundamental form of M (Duggal & Bejancu 1996). Denoting the connection induced on the lightlike hypersurface from the semi-symmetric metric connection $\tilde{\nabla}$ by ∇, we have:

$$\tilde{\nabla}_{\phi X} \phi Y = \phi (\nabla_X Y) + m(X, Y) N \tag{8}$$

for arbitrary vector fields X and Y on M, where m is a tensor of type (0,2) of the lightlike hypersurface M and we call (8) the *Gauss formula* with respect to the induced connection ∇.

Using (5), we get:

$$\tilde{\nabla}_{\phi X} \phi Y = \overset{\circ}{\tilde{\nabla}}_{\phi X} \phi Y + \tilde{\pi}(\phi Y) \phi X - \tilde{g}(\phi X, \phi Y) \tilde{Q},$$

and hence, combining (7) and (8), we can also write:

$$\begin{aligned} \phi(\nabla_X Y) + m(X, Y)N = \; & \phi(\overset{\circ}{\nabla}_X Y) + B(X, Y)N \\ & + \tilde{\pi}(\phi Y)\phi X - \tilde{g}(\phi X, \phi Y)\tilde{Q}. \end{aligned} \tag{9}$$

Writing the value of $\tilde{Q}$ given by (6) in (9), we have:

$$\phi(\nabla_X Y) + m(X, Y)N = \phi(\overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X, Y)Q) + \{B(X, Y) - \mu g(X, Y)\}N.$$

With respect to this equation, we obtain:

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X, Y)\, Q, \tag{10}$$

where $\pi(X) = \tilde{\pi}(\phi X)$ and

$$m(X, Y) = B(X, Y) - \mu g(X, Y). \tag{11}$$

Using (10) and the fact that the connection induced on lightlike hypersurface from Levi-Civita connection is not a metric connection, we get:

$$(\nabla_X g)(Y, Z) = \{m(X, Y) + \mu g(X, Y)\}\eta(Z) + \{m(X, Z) + \mu g(X, Z)\}\eta(Y) \tag{12}$$

where $\eta(Z) = \tilde{g}(Z, N)$.

By (10), we also have:

$$T(X, Y) = \pi(Y)X - \pi(X)Y. \tag{13}$$

Combining (12) and (13), we have:

Proposition 1. *The connection induced on a lightlike hypersurface of a semi-Riemannian manifold with a semi-symmetric metric connection is semi-symmetric, but not a metric connection.*

Here, we note that the induced connection ∇ is a metric connection if the lightlike hypersurface is totally geodesic with respect to a semi-symmetric metric connection $\tilde{\nabla}$ and $\mu = 0$.

The Weingarten formula with respect to the Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$ is:

$$\overset{\circ}{\tilde{\nabla}}_{\phi X} N = -\phi(\underset{N}{\overset{\circ}{A}} X) + \tau(X)N \tag{14}$$

for any vector field X on M, where $\underset{N}{\overset{\circ}{A}}$ is the shape operator of M and τ is the 1-*form* (Duggal & Bejancu 1996).

On the other hand, denoting the projection of TM on $S(TM)$ with respect to the decomposition (1) by P and using (5), we get:

$$\tilde{\nabla}_{\phi X} N = \overset{\circ}{\tilde{\nabla}}_{\phi X} N + \lambda\phi X - \eta(X)\phi Q - \mu\eta(X)N,$$

since

$$\tilde{\pi}(N) = \tilde{g}(\tilde{Q}, N) = \tilde{g}(P\phi Q + \lambda\xi + \mu N, N) = \lambda g(\xi, N) = \lambda.$$

Therefore, using (14), we see that:

$$\tilde{\nabla}_{\phi X} N = -\phi(A_N X + \eta(X)Q) + (\tau(X) - \eta(X))N, \tag{15}$$

where $A_N = \overset{\circ}{\underset{N}{A}} - \lambda I$ and I is the unit tensor. Equation (15) is called the *Weingarten formula* with respect to the semi-symmetric connection.

Proposition 2. *Let S(TM) and $S(TM)'$ be two screen distributions on M and m and m′ be the second fundamental forms with respect to tr(TM) and $tr(TM)'$, respectively. Then $m = m'$ on U that is the local second fundamental form of M on U with respect to a semi-symmetric connection ∇ is independent of the choice of screen distribution.*

Proof. The proof follows from (8) for both screen distributions *S(TM)* and $S(TM)'$. In fact, we have:

$$\begin{aligned} m(X, Y) &= \tilde{g}(\tilde{\nabla}_{\phi X}\phi Y - \phi(\nabla_X Y), \xi) \\ &= \tilde{g}(\tilde{\nabla}_{\phi X}\phi\, Y, \xi) - g(\phi(\nabla_X Y), \xi) \\ &= \tilde{g}(\tilde{\nabla}_{\phi X}\phi\, Y, \xi) \\ &= m'(X, Y) \end{aligned}$$

for any vector fields X and Y on M.

Thus we have:

Corollary 1. *The second fundamental form with respect to a semi-symmetric connection of a lightlike hypersurface is degenerate.*

Since we denote the projection of *TM* on *S(TM)* with respect the decomposition (1) by P we have:

$$\overset{\circ}{\underset{\phi X}{\nabla}} P\phi\, Y = \phi\,(\overset{\circ\,*}{\underset{X}{\nabla}} PY) + C(X, PY)\xi \tag{16}$$

and

$$\overset{\circ}{\underset{\phi X}{\nabla}} \xi = -\phi(\overset{\circ\,*}{\underset{\xi}{A}} X) - \tau(X)\xi \tag{17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$, where $\phi(\overset{\circ\,*}{\underset{X}{\nabla}} PY)$ and $\phi(\overset{\circ\,*}{\underset{\xi}{A}} X)$ belong to $\Gamma(S(TM))$ and C is *1-form* on M such that C is defined by:

$$C(X, PY) = g(\overset{\circ}{\underset{N}{A}} X, PY) \tag{18}$$

(Duggal & Bejancu 1996). Taking care of the equation (16), we have:

$$\nabla_{\phi X} P\phi Y = \phi(\overset{*}{\underset{X}{\nabla}} PY) + D(X, PY)\xi, \tag{19}$$

where $\phi(\overset{*}{\underset{X}{\nabla}} PY)$ belongs to $\Gamma(S(TM))$ and D is *1-form* on M.

From (10), we obtain:

$$\nabla_{\phi X} P\phi\, Y = \overset{\circ}{\underset{\phi X}{\nabla}} P\phi\, Y + \pi(P\phi\, Y)\,\phi X - g(\phi\, X, P\phi Y)\,\phi\, Q,$$

where ϕQ is a vector field defined by: $\phi Q = PQ + \lambda\xi$.

Using (16) and (19), we have:

$$\begin{aligned}\phi(\overset{*}{\underset{X}{\nabla}} PY) + D(X, PY)\xi = \quad & \phi(\overset{\circ}{\overset{*}{\underset{X}{\nabla}}} PY) + C(X, PY)\xi \\ & + \pi(P\phi Y)\phi X - g(X, PY)(P\phi Q + \lambda\xi)\end{aligned}$$

from which:

$$D(X, PY) = C(X, PY) - \lambda g(X, PY) \tag{20}$$

and

$$\overset{*}{\underset{X}{\nabla}} PY = \overset{\circ}{\overset{*}{\underset{X}{\nabla}}} PY + \pi(PY)X - g(X, PY)PQ \tag{21}$$

for any X, $Y \in \Gamma(TM)$.

By using (21), we find:

$$\overset{*}{\underset{X}{\nabla}}(g(PY, PZ)) = (\overset{*}{\underset{X}{\nabla}} g)(PY, PZ) + \overset{\circ}{\overset{*}{\underset{X}{\nabla}}}(g(PY, PZ)) - (\overset{\circ}{\overset{*}{\underset{X}{\nabla}}} g)(PY, PZ).$$

From this equation, we get:

$$(\overset{*}{\underset{X}{\nabla}} g)(PY, PZ) = 0. \tag{22}$$

From (21), we also have:

$$\overset{*}{T}(PX, PY) = \pi(PY)PX - \pi(PX)PY. \tag{23}$$

Combining (22) and (23), we have:

Proposition 3. *The connection* $\overset{*}{\nabla}$ *induced on a screen distribution of lightlike hypersurface is a semi-symmetric metric connection.*

EQUATIONS OF GAUSS AND CODAZZI

We denote the curvature tensor of $\tilde{M}$ with respect to the Levi-Civita connection $\overset{\circ}{\tilde{\nabla}}$ by:

$$\overset{\circ}{\tilde{R}}(\tilde{X},\tilde{Y})\tilde{Z} = \overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\overset{\circ}{\tilde{\nabla}}_{\tilde{Y}}\tilde{Z} - \overset{\circ}{\tilde{\nabla}}_{\tilde{Y}}\overset{\circ}{\tilde{\nabla}}_{\tilde{X}}\tilde{Z} - \overset{\circ}{\tilde{\nabla}}_{[\tilde{X},\tilde{Y}]}\tilde{Z},$$

and the curvature tensor of M with respect to induced connection $\overset{\circ}{\nabla}$ by:

$$\overset{\circ}{R}(X,Y)Z = \overset{\circ}{\nabla}_X\overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y\overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X,Y]} Z.$$

The Gauss-Codazzi equations of the lightlike hypersurface are given by:

$$\begin{aligned}\tilde{g}(\overset{\circ}{\tilde{R}}(\phi X,\phi Y)\phi Z, P\phi W) = \ & g(\overset{\circ}{R}(X,Y)Z, PW) \\ & + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW),\end{aligned} \tag{24}$$

$$\begin{aligned}\tilde{g}(\overset{\circ}{\tilde{R}}(\phi X,\phi Y)\phi Z,\xi) = \ & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + B(Y,Z)\tau(X) - B(X,Z)\tau(Y),\end{aligned} \tag{25}$$

and

$$\tilde{g}(\overset{\circ}{\tilde{R}}(\phi X,\phi Y)\phi Z, N) = \tilde{g}(\overset{\circ}{R}(\phi X,\phi Y)\phi Z, N) \tag{26}$$

for any $X, Y, Z, W \in \Gamma$ (TM) (Duggal & Bejancu 1996).

Now we shall find the equations of Gauss and Codazzi of the lightlike hypersurface with a semi-symmetric connection. The curvature tensor of the semi-symmetric metric connection $\tilde{\nabla}$ of $\tilde{M}$ is, by definition,

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}.$$

Writing $\tilde{X} = \phi X$, $\tilde{Y} = \phi Y$, $\tilde{Z} = \phi Z$, we obtain:

$$\tilde{R}(\phi X,\phi Y)\phi Z = \tilde{\nabla}_{\phi X}\tilde{\nabla}_{\phi Y}\phi Z - \tilde{\nabla}_{\phi Y}\tilde{\nabla}_{\phi X}\phi Z - \tilde{\nabla}_{\phi[X,Y]}\phi Z.$$

Thus, using (8) and (14), we have:

$$\begin{aligned}\tilde{R}(\phi X,\phi Y)\phi Z = \ & \phi(R(X,Y)Z)+m(X,Z)(A_N Y+\eta(Y)Q)-m(Y,Z)(A_N X+\eta(X)Q)\\ &+\{m(\pi(Y)X-\pi(X)Y,Z)\ +\ (\nabla_X m)(Y,Z)-(\nabla_Y m)(X,Z)\\ &+m(Y,Z)(\tau(X)-\mu\eta(X))-m(X,Z)(\tau(Y)-\mu\eta(Y))\}N\end{aligned} \tag{27}$$

where $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature tensor of the lightlike hypersurface with a semi-symmetric connection ∇.

Then, we combining (14), (18), (20) and (27) we have:

$$\begin{aligned}\tilde{g}(\tilde{R}(\phi X,\phi Y)\phi Z,\phi PW) = \ & g(R(X,Y)Z,PW)+m(X,Z)D(Y,PW)\\ &-m(Y,Z)D(X,PW)\\ &+\{m(X,Z)\eta(Y)-m(Y,Z)\eta(X)\}\pi(PW),\end{aligned} \tag{28}$$

$$\begin{aligned}\tilde{g}(\tilde{R}(\phi X,\phi Y)\phi Z,\xi) = \ & \pi(Y)m(X,Z)-\pi(X)m(Y,Z)+(\nabla_X m)(Y,Z)-(\nabla_Y m)(X,Z)\\ &+m(Y,Z)(\tau(X)-\mu\eta(X))-m(X,Z)(\tau(Y)-\mu\eta(Y)),\end{aligned} \tag{29}$$

and

$$\tilde{g}(\tilde{R}(\phi X,\phi Y)\phi Z,N) = \tilde{g}(R(\phi X,\phi Y)\phi Z,N). \tag{30}$$

We call equations (28)-(30) the *equations of Gauss-Codazzi* of the lightlike hypersurface with a semi-symmetric connection.

THE RICCI TENSOR

For $n > 1$, let M be a lightlike hypersurface of an $(n+1)$ -dimensional semi-Riemannian manifold $\tilde{M}$ with a semi-symmetric metric connection. Considering the definition of the Ricci tensor of M with respect to the symmetric connection, we define the Ricci tensor of M with respect to semi-symmetric connection by:

$$Ric(X, Y) = trace\{Z \to R(X,Z)Y\}, \tag{31}$$

for any $X, Y, Z \in \Gamma(TM)$. Thus, the Ricci tensor of M with respect to a semi-symmetric connection is given by:

$$Ric(X, Y) = \sum_{i=1}^{n-1} \varepsilon_i g(R(X, W_i)Y, W_i) + \tilde{g}(R(X,\xi)Y,N) \tag{32}$$

where $\{W_1, ..., W_{n-1}\}$ is an orthonormal basis of the screen distribution. Hence, by using (10) and (32) we get:

$$Ric(X, Y) - Ric(Y,X) = 2d\tau(X, Y) + (n-2)d\pi(X, Y) \tag{33}$$

for any $\forall X, Y \in \Gamma(TM)$.

From (33) we have:

Proposition 4. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\tilde{M}$ with a semi-symmetric metric connection. Then the Ricci tensor of a lightlike hypersurface with respect to a semi-symmetric connection is symmetric if and only if the 1-forms τ and π are closed.*

Now suppose that the *1-form* π is closed. In this case we can define the sectional curvature for a section in $\tilde{M}$ with respect to the semi-symmetric metric connection (Imai 1972$_a$).

We assume that the semi-symmetric metric connection $\tilde{\nabla}$ is of constant sectional curvature, then $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}$ should be of the form

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = c\{\tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y} - \tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X}\} \tag{34}$$

for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(T\tilde{M})$, where c is a certain scalar. Thus, $\tilde{M}$ is a semi-Riemannian manifold of constant curvature c with respect to a semi-symmetric metric connection and is denoted by $\tilde{M}(c)$.

Proposition 5. *Let M be a lightlike hypersurface of an $(n+1)$-dimensional semi-Riemannian space form $\tilde{M}(c)$ with a semi-symmetric metric connection. Then we have:*

$$\begin{aligned} Ric(X, Y) = \quad & (n-1)cg(X, Y) - m(X, Y)H + \sum_{i=1}^{n-1} \varepsilon_i m(W_i, Y)D(X, W_i) \\ & + \eta(X) \sum_{i=1}^{n-1} \varepsilon_i m(W_i, Y)g(W_i, Q) \end{aligned} \tag{35}$$

for any $X, Y, Z \in \Gamma(TM)$, where H is the mean curvature of M with respect to a semi-symmetric connection, given by:

$$H = \sum_{i=1}^{n-1} \varepsilon_i g(A_N W_i, W_i).$$

Proof. Using (27) in (32) and considering (34), we obtain (35).

Proposition 6. *Let M be a lightlike hypersurface of an $(n+1)$-dimensional semi-Riemannian space form $\tilde{M}$ (c) with a semi-symmetric metric connection. If M is totally geodesic with respect to a semi-symmetric metric connection, then the Ricci tensor of M with respect to semi-symmetric connection is symmetric.*

Proof. The proof is an obvious consequence of (35).

Proposition 7. *Let M be a lightlike hypersurface of an $(n+1)$-dimensional semi-Riemannian space form $\tilde{M}(c)$ with a semi-symmetric metric connection. If M is totally geodesic with respect to a semi-symmetric metric connection and $\mu = 0$ then the Ricci tensor of M is parallel with respect to a semi-symmetric connection ∇.*

Proof. First of all, we compute the derivative of the Ricci tensor of M with respect to a semi-symmetric connection. We define:

$$(\nabla_Z Ric)(X, Y) = \nabla_Z Ric(X, Y) - Ric(\nabla_Z X, Y) - Ric(X, \nabla_Z Y).$$

Then, from the equations (35) and (12) we obtain:

$$\begin{aligned}(\nabla_Z Ric)(X, Y) = \; & (n-1)c(\nabla_Z g)(X, Y) - (\nabla_Z m)(X, Y)H - m(X, Y)Z(H) \\ & + \sum_{i=1}^{n-1} \varepsilon_i \{(\nabla_Z m)(W_i, Y)D(X, W_i) + m(\nabla_Z W_i, Y)D(X, W_i) \\ & + m(W_i, Y)(\nabla_Z D)(X, W_i) + m(W_i, Y)D(X, \nabla_Z W_i)\} \\ & + (\tau(Z)\eta(X) - g(A_N Z, X) - \eta(Z)\pi(X)) \sum_{i=1}^{n-1} \varepsilon_i m(W_i, Y)\pi(W_i) \\ & + \eta(X) \sum_{i=1}^{n-1} \varepsilon_i \{[(\nabla_Z m)(W_i, Y) + m(\nabla_Z W_i, Y)]\pi(W_i) \\ & + m(W_i, Q)([m(Z, W_i) + \mu g(Z, W_i)]\eta(Q) + g(\nabla_Z W_i, Q) + g(W_i, \nabla_Z Q))\}.\end{aligned} \tag{36}$$

Hence, we get that if M is totally geodesic with respect to a semi-symmetric metric connection and $\mu = 0$, then the right hand side of the equation (36) is vanishing for any $X, Y, Z \in \Gamma(TM)$. Therefore, the proof is completed.

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مواءمة الفوسطوح مثيل المنطو الريماني
مع مثيل الارتباط المتري المتناظر

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خلاصة

تبين هذه الدراسة أن الارتباط المستنتج على مواءمة الفوسطوح مثيل المنطو الريماني مع مثيل الارتباط المتري المتناظر مثيل متناظر، ولكن لا يكون ارتباط متري ونحصل على معادلات جاوس وكودازي. ثم ندرس الشروط التي تجعل موتر ريكاتي متناظر ومتوازي.