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On contact metric hypersurfaces in a real space form

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ABSTRACT

For a $(2n+1)$-dimensional $N(k)$-contact metric hypersurface in a real space form $\tilde{M}(c)$, some main results are obtained as follows: **(1)** if $k-c>0$ then M is totally umbilical, and consequently, either M is a Sasakian manifold of constant curvature $+1$ or M is 3-dimensional and flat; **(2)** if $k=c$ and M is Einstein then either M is totally geodesic or a developable hypersurface in $\tilde{M}(k)$, in particular M is of constant curvature and consequently, either M is a Sasakian manifold of constant curvature $+1$ or M is 3-dimensional and flat; **(3)** if M is 3-dimensional non-Sasakian such that $k=c$ then either M is flat or the shape operator of M is of a specific form (see Theorem 6); and **(4)** if M is η-Einstein such that $n \geq 2$ and $k=c$, then M is a developable hypersurface. An obstruction for M to be totally geodesic is also obtained.

Keywords: $(k\,,\,\mu)$-manifold; $N(k)$-contact metric manifold; $N(k)$-contact metric hypersurface; developable hypersurface; Einstein manifold; η-Einstein manifold.

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INTRODUCTION

In contact geometry, Takahashi (1969) studied Sasakian hypersurfaces in a real space form and proved that a pseudo-Sasakian manifold M^{2n+1} isometrically immersed in a pseudo-Riemannian manifold $\tilde{M}^{2n+2}(c)$ of constant curvature $c \neq 1$ has constant curvature $+1$. In the case when $c=1$, M^{2n+1} also has constant curvature $+1$ if M^{2n+1} is η-Einstein. Takahashi and Tanno (1971) studied a K-contact Riemannian manifold M^{2n+1} isometrically immersed in a real space form $\tilde{M}^{2n+2}(c)$ and proved that: **(1)** if the ambient space is of constant curvature $c=1$, then the immersed hypersurface is Sasakian, and **(2)** if the ambient space is of constant curvature $c \neq 1$, then $c<1$ and the immersed hypersurface is of constant curvature $+1$ and hence also Sasakian. Blair, Koufogiorgos and Sharma (1990) proved that on a 3-dimensional contact metric manifold M equipped with a contact metric structure (φ, ξ, η, g), the following conditions are equivalent: **(1)** M is η-Einstein, **(2)** $Q\varphi = \varphi Q$, where Q is the Ricci operator, and **(3)** ξ belongs to the k-nullity distribution, that is, $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y$, where R is the curvature tensor and k is a constant ≤ 1. A 3-dimensional contact metric manifold M satisfying one of the conditions (1), (2) or (3) is either Sasakian, flat or of constant ξ-sectional curvature $k < 1$ and constant φ-sectional curvature $-k$. Finally, they showed that if such an M is isometrically immersed in a 4- dimensional manifold of constant sectional curvature $+1$, then M is Sasakian. In De, Shaikh and Biswas (2003), this result was generalized to higher dimension and proved that if M^{2n+1} is a contact metric manifold with $n > 1$, such that ξ belongs to the k-nullity distribution, isometrically immersed in a real space form $\tilde{M}^{2n+2}(1)$ of constant curvature $+1$, then M^{2n+1} is Sasakian.

A contact metric manifold in which the structure vector field belongs to the k-nullity distribution is known as $N(k)$-contact metric manifold (Blair *et al.* 2005). Motivated by the studies of Sasakian, K-contact and $N(k)$-contact metric hypersurfaces in real space form, in the present paper we investigate contact metric hypersurfaces in a real space form. In section 2, a brief account of contact metric manifolds, (k, μ)-manifold, and $N(k)$-contact metric manifolds are presented. In section 3, we prepare some basic equations for $N(k)$-contact metric hypersurfaces in a real space form. As the main result of this section we prove that if M is a $(2n+1)$-dimensional $N(k)$-contact metric hypersurface in a real space form $\tilde{M}(c)$ such that $k - c > 0$, then M is totally umbilical, and consequently, either M is a Sasakian manifold of constant curvature $+1$ or M is 3-dimensional and flat. In section 4, Einstein $N(k)$-contact metric hypersurfaces in a real space form are studied. For an $N(k)$-contact metric hypersurface M in a real space form $\tilde{M}(c)$ we prove that: **(a)** if M is Einstein and $k = c$, then either M is totally geodesic or a developable hypersurface in $\tilde{M}(c)$, in particular M is of constant curvature and consequently, either M is a Sasakian manifold of constant curvature $+1$ or M is 3-dimensional and flat; and **(b)** if M is totally geodesic, then $c = k$. In the last section 5, η-Einstein $N(k)$-contact metric hypersurfaces are studied. It is proved that if M is a 3-dimensional non-Sasakian η-Einstein $N(k)$-contact metric hypersurface in a real space form $\tilde{M}^4(k)$, then either M is flat or the shape operator A of M takes the form:

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\frac{2k}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

such that $\lambda \neq -\frac{2k}{\lambda}$. Finally, it is shown that if M is a $(2n+1)$-dimensional η-Einstein $N(k)$-contact metric hypersurface in a real space form $\tilde{M}(c)$ such that $2n+1 \geq 5$ and $k = c$, then M is a developable hypersurface in $\tilde{M}(c)$.

CONTACT METRIC MANIFOLDS

A differentiable 1-form η on a $(2n+1)$-dimensional differentiable manifold M is called a *contact form* if $\eta \wedge (d\eta)^n$ is non-vanishing everywhere on M, and M equipped with a contact form is a *contact manifold*. On a contact manifold there exists a unique global vector field ξ called the characteristic vector field, such that:

$$\eta(\xi) = 1 \qquad \text{and} \qquad \mathrm{d}\eta(\xi\,,\,\cdot) = 0. \tag{1}$$

Moreover, there exists a $(1\,,\,1)$-tensor field φ and a Riemannian metric g such that:

$$\varphi(\xi) = 0, \qquad \eta \circ \varphi = 0, \qquad \eta(X) = g(X, \xi), \tag{2}$$

$$\varphi^2 = -I + \eta \otimes \xi, \qquad d\eta(X, Y) = g(X, \varphi Y), \tag{3}$$

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{4}$$

for $X, Y \in TM$. The structure (φ, ξ, η, g) is called a *contact metric structure* and the manifold M endowed with such a structure is said to be a *contact metric manifold*. In a contact metric manifold M, the $(1\,,\,1)$-tensor field h, defined by half of the Lie derivative of φ in the direction ξ, is symmetric and satisfies:

$$h\xi = 0, \qquad h\varphi + \varphi h = 0, \qquad \nabla\xi = -\varphi - \varphi h \text{ and } \mathrm{trace}(h) = \mathrm{trace}\,(\varphi h) = 0\,.$$

A contact metric structure (φ, ξ, η, g) on M gives rise to a natural almost Hermitian structure on the product manifold $M \times \mathrm{R}$. If this structure is integrable, then M is said to be a *Sasakian manifold*. A Sasakian manifold is characterized by the condition:

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection. A contact metric manifold is called a *K-contact manifold* if the structure vector field ξ is a Killing vector field.

A $(2n+1)$-dimensional contact metric manifold M is a $(k\,,\,\mu)$-manifold (Blair *et al.* 1995) if it satisfies the $(k\,,\,\mu)$-nullity condition, that is:

$$R(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \qquad X, Y \in TM$$

for k, μ being real constants. If $\mu = 0$, it is an $N(k)$-contact metric manifold (Blair *et al.* 2005). On a (k, μ)-manifold, it follows that $k \leq 1$. For a (k, μ)-manifold, the conditions of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent. The standard contact metric structure on the unit tangent sphere bundle satisfies the $(c(2-c), -2c)$-nullity condition if and only if the base manifold M is of constant curvature c. Characteristic examples of non-Sasakian (k, μ)-manifolds are the unit tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups. We also recall the notion of a D_a- homothetic deformation. For a given contact metric structure (φ, ξ, η, g), this is the structure defined by:

$$\bar{\eta} = a\eta,\ \bar{\xi} = \frac{1}{a}\xi,\ \bar{\varphi} = \varphi,\ \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric, K-contact or Sasakian, it destroys a condition like $R(X,Y)\xi = 0$ or $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$. However the form of the (k, μ)-nullity condition is preserved under a D_a-homothetic deformation with:

$$\overline{k} = \frac{k + a^2 - 1}{a^2} \qquad \text{and} \qquad \overline{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (k, μ)-manifold M, Boeckx (2000) introduced an invariant $I_M = \frac{1-\mu/2}{\sqrt{1-k}}$ and showed that two non-Sasakian (k, μ)-manifolds M_1 and M_2 are locally isometric as contact metric manifolds up to a D_a-homothetic deformation if and only if $I_{M_1} = I_{M_2}$. For more details we refer to Baikoussis *et al.* (1992), Blair (2002), Blair, *et al.* (1995) and Boeckx (2000).

CONTACT METRIC HYPERSURFACES

A Riemannian manifold with constant sectional curvature c, denoted by $M(c)$, is called a *real space form*. The model spaces for real space forms are the Euclidean spaces $(c = 0)$, the spheres $(c > 0)$ and the hyperbolic spaces $(c < 0)$. Let (M, g) be an isometrically immersed hypersurface in a real space form $\tilde{M}(c)$. Then the Gauss and Codazzi equations are given by:

$$R(X,Y)Z = c(X \wedge Y)Z + (AX \wedge AY)Z, \tag{5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = 0, \tag{6}$$

respectively, where R is the curvature tensor of M, A is the field of second fundamental operators, ∇ is the induced connection on M, and $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$.

A hypersurface, which is a contact metric manifold or an $N(k)$-contact metric manifold will be called a *contact metric hypersurface* or an $N(k)$-*contact metric hypersurface*, respectively.

Example 1. A $(2n+1)$-dimensional $N(0)-$contact metric manifold is locally isometric to the trivial sphere bundle $E^{n+1} \times S^n(4)$, which is well known to be a real hypersurface of E^{2n+2} (a real space form of zero curvature).

Example 2. A $(2n+1)$-dimensional sphere S^{2n+1} in E^{2n+2} is a $N(1)$-contact metric, that is, a Sasakian manifold.

Example 3. (Example 9.1, Chen & Mihai 2005) Consider the cylindrical hypersurface

$$f : M \equiv \mathrm{R} \times S^2(1) \to E^4$$

defined by:

$$f(t,\theta,\phi) = (t,\ \cos\theta\cos\phi,\ \sin\theta\sin\phi,\ \sin\phi), \tag{7}$$

where E^4 is the Euclidean 4-space endowed with the flat Riemannian metric:

$$g = dt^2 + d\phi^2 + \cos^2\phi d\theta^2. \tag{8}$$

Define an almost contact metric structure (φ, ξ, η, g) on M by:

$$\eta = \cos\theta dt + \sin\theta d\phi, \xi = \cos\theta\frac{\partial}{\partial t} + \sin\theta\frac{\partial}{\partial\phi},$$

$$\varphi\left(\frac{\partial}{\partial t}\right) = -\tan\theta\frac{\partial}{\partial\theta}, \qquad \varphi\left(\frac{\partial}{\partial\phi}\right) = \frac{\partial}{\partial\theta}, \text{and} \tag{9}$$

$$\varphi\left(\frac{\partial}{\partial\theta}\right) = \cos\phi\left(\sin\theta\frac{\partial}{\partial t} - \cos\theta\frac{\partial}{\partial\phi}\right).$$

Consider the orthonormal frame $\{e_1, e_2, e_3\}$ on M given by:

$$e_1 = \xi, e_2 = -\sin\theta \frac{\partial}{\partial t} + \cos\theta \frac{\partial}{\partial \phi} \text{ and } e_3 = \sec\phi \frac{\partial}{\partial \theta}.$$

With respect to this frame, we have:

$$\varphi e_1 = 0, \qquad \varphi e_2 = e_3 \text{ and } \varphi e_3 = -e_2 .$$

It is easy to verify that g satisfies:

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ X, Y \in TM.$$

Moreover, on M we also have:

$$\eta \wedge d\eta = dt \wedge d\theta \wedge d\phi \neq 0.$$

So $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold. This contact metric hypersurface $(M, \varphi, \xi, \eta, g)$ in E^4 is non-K-contact.

Example 4. (Example 3.1, Blair *et al.* 2005) Consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature $\frac{(\sqrt{n}+1)^2}{n-1}$. After the D_a-homothetic deformation, where $a = \frac{(\sqrt{n}+1)^2}{n-1} + 1$, we obtain an $N\left(1 - \frac{1}{n}\right)$-contact metric manifold. For $n = 2$, it can be topologically imbedded in the sphere S^6.

In an $N(k)$-contact metric manifold, the curvature tensor R satisfies Blair (2002):

$$R(\xi, X)Y = k(g(X, Y)\xi - \eta(Y)X), \ X, Y \in TM. \tag{10}$$

Now, let M be an $N(k)$-contact metric hypersurface. From (5) and (10) we have:

$$(k - c)\{g(X, Y)\xi - \eta(Y)X\} - g(AX, Y)A\xi + \eta(AY)AX = 0 \tag{11}$$

for all $X, Y \in TM$. Let $\lambda_1, ..., \lambda_{2n+1}$ be the principal curvatures and let $e_1, ..., e_{2n+1}$ be the principal directions, that is:

$$Ae_i = \lambda_i e_i , \qquad i \in \{1, \ldots, 2n+1\} . \tag{12}$$

From (11) and (12) we get:

$$(k - c)\{g(e_i, e_j)\xi - \eta(e_j)e_i\} - \lambda_i g(e_i, e_j)A\xi + \lambda_i\lambda_j\eta(e_j)e_i = 0, \tag{13}$$

where $i,j \in \{1, \ldots, 2n+1\}$.

Now, we prove the following:

Proposition 1. Let M be a $(2n+1)$-dimensional $N(k)$-contact metric hypersurface in a real space form $\tilde{M}(c)$. If $k=c$ and $rank\, A \geq 2$ at a point $x \in M$, then $A\xi_x = 0$.

Proof. Since $k = c$, from (13) we get:

$$\lambda_i \lambda_j \eta(e_j) e_i = \lambda_i (e_i, e_j) A\xi_x , \qquad i,j \in \{1, \ldots, 2n+1\} . \tag{14}$$

Since $rankA \geq 2$ is assumed, we may suppose that $\lambda_1\lambda_2 \neq 0$. Then from the above equation, we have:

$$\lambda_1\lambda_2\eta(e_2)e_1 = \lambda_1 g(e_1, e_2)A\xi_x = 0,$$

which shows that $\eta(e_2) = 0$. Hence, putting $i = j = 2$ in (12) we get:

$$0 = (\lambda_2)^2 \eta(e_2)e_2 = \lambda_2 A\xi_x = 0$$

which completes the proof.

We need the following lemma:

Lemma 1. Let M be a $(2n+1)$-dimensional contact metric hypersurface in a real space form $\tilde{M}(c)$. Let ξ be a principal direction of A corresponding to the principal curvature μ and let λ_1 be a principal curvature of multiplicity m_1, such that $\lambda_1 \neq \mu$. Then $m_1 \leq n$.

Proof. Let there be principal curvatures $\lambda_1, ..., \lambda_k$ of multiplicity $m_1, \ldots, m_k$, respectively. Let the corresponding eigenspaces be denoted by $D_{\lambda_1}, ..., D_{\lambda_k}$. Then we may write:

$$TM = D_{\lambda_1} \oplus D_{\lambda_2} \oplus ... \oplus D_{\lambda_k},$$

such that $\xi \in D_\mu$ for some $\mu \in \{\lambda_2, ..., \lambda_k\}$. Now, let $X, Y \in D_{\lambda_1}$. Then in view of (6) we get:

$$\begin{aligned} 0 &= (\nabla_X AY - A\nabla_X Y) - (\nabla_Y AX - A\nabla_Y X) \\ &= \nabla_X(\lambda_1 Y) - A\nabla_X Y - \nabla_Y(\lambda_1 X) + A\nabla_Y X \\ &= -A[X, Y] + \lambda_1 \nabla_X Y + (X\lambda_1)Y - \lambda_1 \nabla_Y X - (Y\lambda_1)X, \end{aligned}$$

which implies that:

$$(A - \lambda_1 I)[X, Y] = (X\lambda_1)Y - (Y\lambda_1)X \in D_{\lambda_1}.$$

If $Z \in D_\alpha, \alpha \in \{\lambda_2, ..., \lambda_k\}$ then:

$$\begin{aligned}\lambda_1 g([X, Y], Z) &= g(A[X, Y] - (X\lambda_1)Y + (Y\lambda_1)X, Z) \\ &= g([X, Y], AZ) = g([X, Y], \alpha Z) = \alpha g([X, Y], Z).\end{aligned}$$

Since $\lambda_1 \neq \alpha$, from the above equation we conclude that D_{λ_1} is involutive. Moreover $\eta([X, Y]) = 0$. Since the dimension of a submanifold, orthogonal to the structure vector field ξ, in a $(2n + 1)$-dimensional contact metric manifold is at most n (Blair (2002), p. 55), therefore $m_1 \leq n$.

We also recall the following:

Theorem 1. (Blair (2002), pp. 98-99). A contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat.

Now we give the main result of this section as follows.

Theorem 2. Let M be a $(2n+1)$-dimensional $N(k)$-contact metric hypersurface in a real space form $\tilde{M}(c)$ such that $k-c > 0$. Then M is totally umbilical, and consequently, either M is a Sasakian manifold of constant curvature $+1$ or M is 3-dimensional and flat.

Proof. Let $\lambda_1, ..., \lambda_{2n+1}$ be the principal curvatures and let $e_1, ..., e_{2n+1}$ be the corresponding principal directions. From (13) we get:

$$\left(\lambda_i\lambda_j - k + c\right)\eta(e_j)e_i = 0, \;\; i \neq j, \text{ and} \tag{15}$$

$$(k - c)\{\xi - \eta(e_i)e_i\} - \lambda_i A\xi + (\lambda_i)^2\eta(e_i)e_i = 0 \tag{16}$$

for all $i, j \in \{1,... ,2n + 1\}$. From (15) we have two cases. In the first case,

$$\lambda_i\lambda_j - k + c = 0, \;\; i \neq j. \tag{17}$$

In the second case,

$$\eta(e_j) = 0, \;\; \text{for some } j. \tag{18}$$

Since $k > c$ is assumed, therefore from (17) it follows that the principal curvatures are all equal, that is:

$$\lambda_1 = \lambda_1 = \cdots = \lambda_{2n+1} = \lambda \text{ and } \lambda^2 = k - c\,, \tag{19}$$

which shows that M is totally umbilical. In the second case, let e_j be a principal vector such that $\eta(e_j) = 0$. Then from (16) we get:

$$A\xi = \frac{k-c}{\lambda_j}\xi.$$

From the previous equation we may assume that $e_{2n+1} = \xi$. Then we have:

$$\eta(e_i) = 0, \; i \in \{1, ..., 2n\}$$

and consequently from (16) we get:

$$(k-c)\xi = \lambda_i \lambda_{2n+1}\xi, \; i \in \{1, ..., 2n\}. \tag{20}$$

Since $k > c$ is assumed, therefore from (20) we have:

$$\lambda_1 = \lambda_2 = ... = \lambda_{2n} = \lambda \neq 0 \text{ and } \lambda_{2n+1} = \frac{k-c}{\lambda}, \tag{21}$$

which in view of Lemma 1 yields (19). Hence, again M is totally umbilical. Using the condition $\lambda^2 = k - c$ of (19) in (5), we get:

$$R(X, Y) = (c + \lambda^2)(X \wedge Y) = k\,(X \wedge Y)\,,$$

that is, M is of constant curvature k; which in view of Theorem 1 completes the proof.

EINSTEIN CONTACT METRIC HYPERSURFACES

Let M be an n-dimensional Riemannian manifold. The *Ricci tensor* S is defined by:

$$S(X, Y) = \sum_{j=1}^{n} R(e_j, X, Y, e_j), X, Y \in TM,$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis for T_pM. The *Ricci curvature of* X, denoted $Ric(X)$, is defined by $Ric(X) = S(X, X)$, and the Ricci curvatures determine the Ricci tensor completely.

Now, we recall the following:

Theorem 3. (Theorem 4, Chen (1999)) Let M be an n-dimensional submanifold in a real space form $\tilde{M}(c)$. Then, the squared mean curvature satisfies:

$$\|H\|^2 \geq \frac{4}{n^2}\{Ric(X) - (n-1)c\} \tag{22}$$

for each unit vector $X \in T_pM$. The equality case of (22) holds for all unit vectors $X \in T_pM$ and for all $p \in M$ if and only if either M is a totally geodesic submanifold or M is a totally umbilical surface.

A Riemannian manifold is an Einstein manifold with the Ricci curvature ρ if the Ricci tensor S equals ρg, where g is the Riemannian metric. Every 2-dimensional Riemannian manifold is an Einstein manifold and every 3-dimensional Einstein manifold is a Riemannian manifold of constant curvature. For an Einstein submanifold, we can state the following:

Corollary 1. Let M be an Einstein manifold of dimension $n \geq 3$ with the Ricci curvature ρ, which is isometrically immersed in a real space form $\tilde{M}(c)$ of constant curvature c. Then, the squared mean curvature satisfies:

$$\|H\|^2 \geq \frac{4}{n^2}\{\rho - (n-1)c\}. \tag{23}$$

The equality case of (23) holds if and only if M is totally geodesic.

From the above theorem we see that if M is a totally geodesic Einstein submanifold then the equality case of (23) becomes $\rho = (n-1)c$. But if $\rho = (n-1)c$ is satisfied, then it is not known whether the submanifold is totally geodesic or not. However, in the case of Einstein hypersurfaces the following result is known (see Theorem 7.1 in Fialkow (1938)).

Theorem 4. Let M be a hypersurface of dimension $n \geq 3$ in a real space form $\tilde{M}(c)$ of constant curvature c. If M is an Einstein manifold such that its Ricci curvature ρ satisfies $\rho = (n-1)c$, then M is either a totally geodesic hypersurface or a developable hypersurface (that is, $rank A \leq 1$ at each point of M) in $\tilde{M}(c)$, in particular M is a space of constant curvature c.

Now, we are able to prove the following:

Theorem 5. Let M be a $(2n+1)$-dimensional $N(k)$-contact metric hypersurface in a real space form $\tilde{M}(c)$. Then the following statements are true.

- **(a)** If M is Einstein and $k = c$, then M is either a totally geodesic or a developable hypersurface in $\tilde{M}(c)$, in particular M is of constant curvature and consequently, either M is a Sasakian manifold of constant curvature $+1$ or M is 3-dimensional and flat.
- **(b)** If M is totally geodesic, then c must be k.

Proof. In a $(2n + 1)$-dimensional $N(k)$-contact metric manifold M, the Ricci tensor S satisfies (see Blair 2002):

$$S(X, \xi) = 2nk\eta(X), X \in TM. \tag{24}$$

If M is an Einstein manifold then from (24) it follows that its Ricci curvature ρ satisfies $\rho = 2nk$. If $k = c$, then $\rho = 2nc$. Therefore, from Theorems 1 and 4 we have the statement **(a)**.

Now, we prove the statement **(b)**. If M is totally geodesic then from Theorem 3 we have the equality case of :

$$||H||^2 \geq \frac{4}{(2n+1)^2}\{Ric(X) - 2nX\}$$

for all unit vector $X \in T_pM$ and for all $p \in M$, which is $Ric(X) = 2nc$. In particular, we have $2nc = Ric(\xi) = 2nk$, hence $c = k$.

η-EINSTEIN CONTACT METRIC HYPERSURFACES

A contact metric manifold is said to be η-Einstein (Blair 2002), if the Ricci operator Q satisfies:

$$Q = aI + b\eta \otimes \xi, \tag{25}$$

where a and b are smooth functions on the manifold. The smooth functions a and b will be called the associated scalars. In particular, if $b = 0$ then the manifold is an Einstein manifold.

In a non-Sasakian $N(k)$-contact metric manifold of dimension $2n+1$, the Ricci operator Q is given by (Blair *et al*. 1995):

$$Q = 2(n-1)I + 2(n-1)h + \{2(1-n) + 2nk\}\eta \otimes \xi. \tag{26}$$

Consequently, the Ricci tensor S and the scalar curvature r are given by:

$$S(X, Y) = 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) + \{2(1-n) + 2nk\}\eta(X)\eta(Y), \tag{27}$$

$$r = 2n(2n - -2 + k). \tag{28}$$

From (28), it follows that the scalar curvature of a non-Sasakian $N(k)$-contact metric manifold is constant.

From (25) and (26), we see that a non-Sasakian $N(k)$-contact metric manifold is η-Einstein if and only if it is 3-dimensional. Thus an η-Einstein $N(k)$-contact metric manifold of dimension ≥ 5 must be Sasakian.

Blair and coworkers (1990) proved that a 3-dimensional contact metric manifold is η-Einstein if and only if it is an $N(k)$-contact metric manifold. More precisely, in a 3-dimensional $N(k)$-contact metric manifold, we have:

$$Q = \left(\frac{r}{2} - k\right)I + \left(3k - \frac{r}{2}\right)\eta \otimes \xi. \tag{29}$$

In dimensions ≥ 5 it is known that for any η-Einstein K-contact manifold, a and b are constants (Tanno 1969).

Since in a 3-dimensional Riemannian manifold the Weyl conformal curvature tensor vanishes, therefore it is known that:

$$\begin{aligned} R(X, Y)Z =& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned}$$

where r is the scalar curvature.

Now, let M be a 3-dimensional non-Sasakian $N(k)$-contact metric manifold. From (28) it follows that $r = 2k$ and consequently (29) becomes:

$$Q = 2nk\eta \otimes \xi. \tag{31}$$

Then using (31) in (30), we have

$$\begin{aligned} R(X, Y)Z =& -k(g(Y,Z)X - g(X,Z)Y) + 2k\eta(Z)(\eta(Y)X - \eta(X)Y) \\ & + 2k(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi). \end{aligned} \tag{32}$$

Now, we prove the following theorem:

Theorem 6. Let M be a 3-dimensional non-Sasakian $N(k)$-contact metric hypersurface in a real space form $\tilde{M}^4(k)$. Then either M is flat or the shape operator A of M takes the form:

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\frac{2k}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \lambda^2 \neq -2k. \tag{33}$$

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the principal curvatures and let e_1, e_2, e_3 be the corresponding principal directions, that is,

$$Ae_i = \lambda_i e_i,\ i \in \{1, 2, 3\}. \tag{34}$$

Since $c = k$, from (13) we have:

$$\lambda_i g(e_i, e_j)A\xi - \lambda_i \lambda_j \eta(e_j)e_i = 0,\ i,j \in \{1, 2, 3\}. \tag{35}$$

From (35) we have:

$$\lambda_i A\xi - (\lambda_i)^2 \eta(e_i)e_i = 0,\ i \in \{1, 2, 3\}, \tag{36}$$

$$\lambda_i \lambda_j \eta(e_j)e_i = 0,\ i \neq j. \tag{37}$$

From (32) and (5) we get:

$$\begin{aligned} 0 =& 2k\{g(X,Z)Y - g(Y,Z)X + \eta(Z)(\eta(Y)X - \eta(X)Y) \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\} - g(AY,Z)AX + g(AX,Z)AY \end{aligned} \tag{38}$$

for all $X,Y,Z \in TM$. By taking $X = Z = e_i$ and $Y = e_j$ in (33), in view of (29) we get:

$$0 = \left(2k\left\{1 - \eta(e_i)^2\right\} + \lambda_i \lambda_j\right)e_j + 2k\eta(e_i)\eta(e_j)e_i - 2k\eta(e_j)\xi,\ i \neq j. \tag{39}$$

Now, we have two cases:

Case (a): Let $k = 0$. Then M is flat.

Case (b): Let $k \neq 0$. Then in view of (34) it follows that ξ is a principal direction, say $e_3 = \xi$. Thus we have:

$$\eta(e_1) = 0 = \eta(e_2) \tag{40}$$

and consequently from (37) we get:

$$\lambda_1 \lambda_3 = 0 = \lambda_2 \lambda_3. \tag{41}$$

Using (40) in (39) we also get:

$$\lambda_1 \lambda_2 = -2k. \tag{42}$$

From (42) we see that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, which in view of (41) gives $\lambda_3 = 0$. So in view of Lemma 1 it follows that $\lambda_1 \neq \lambda_2$, and therefore A can be represented by (33).

Now, we recall the following:

Lemma 2. (Lemma 5, Takahashi (1969)) Suppose a Sasakian manifold $M^{2n+1}(\varphi,\xi,\eta,g), n \geq 2$, is isometrically immersed in $\tilde{M}^{2n+2}(1)$. If M^{2n+1} is η-Einstein then $rank A \leq 1$ on M^{2n+1}.

In view of the above lemma and the fact that an η-Einstein $N(k)$-contact metric manifold of dimension ≥ 5 is always Sasakian, we can state the following:

Theorem 7. Let M be a $(2n+1)$-dimensional $N(k)$-contact metric hypersurface, $n \geq 2$, in a real space form $\tilde{M}(c)$. If $k = c$ and M is η-Einstein, then M is a developable hypersurface in $\tilde{M}(c)$.

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حول تلامس مقياس الفوسطحي في شكل الفراغ الحقيقي

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خلاصة

تم التحصل على نتائج رئيسية لتلامس مقياس الفوسطحي في شكل الفراغ الحقيقي ذو $\tilde{M}(c)$ ذو الأبعاد $(2n+1)$ وكانت النتائج كالآتي:

(1) إذا كانت $k - c > 0$ فإن M يكون سُرِّى كامل وبناءً عليه إما M تكون طية ساساكيان ذات تقوس ثابت 1+. أو تكون M ثلاثية الأبعاد ومسطحة.

(2) إذا كان $k = c$ وكانت M اينستان فإنه إما M حيود يزي كامل أو فوسطحي قابل للانبساط في $\tilde{M}(k)$. بالخصوص M تكون ذات تقوس ثابت وبناء عليه إما M تكون طية ساساكانية ذات تقوس ثابت 1+ أو M ثلاثية الأبعاد ومسطحة.

(3) إذا كانت M ثلاثية الأبعاد غير ساساكانية بحيث $k = c$ فإنه إما M تكون مسطحة أو الشكل التأثير لـ M يكون شكل معين (انظر نظرية 6).

(4) إذا كانت M η – اينستانية بحيث أن $\eta \geq 2$ و $k = c$ فإن M تكون فوسطحي قابل للانبساط.

ملاحظة: حول كون M جوديزية كاملة ثم الحصول عليها.