Kuwait J. Sci. Eng. **34** (2A) pp. 41-55, 2007

Tensor Product Surfaces of a Lorentzian Space Curve and a Euclidean Plane Curve

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ABSTRACT

In this paper, we classify all minimal, totally real and complex tensor product surfaces of a Lorentzian space curve and a Euclidean plane curve.

Keywords: Euclidean curve; Lorentzian curve; minimal surface; tensor product.

AMS Subject Classification: 53C50, 53C40.

TENSOR PRODUCT IMMERSIONS

In the Euclidean space E^n, the tensor product immersion of two immersions of a given Riemannian manifold was firstly defined by Chen (1993). In particular, the direct sum and the tensor product maps of two immersions of two different Riemannian manifolds are defined by Decruyenaere et al. (1993) as described here.

Let M and N be two differentiable manifolds and assume that $f : M \to E^m$ and $f : N \to E^n$ are two immersions. The direct sum map and tensor product map are defined respectively by:

$$f \oplus h : M \times N \to E^{m+n} : (p, q) \to (f(p), h(q)) = \left(f^1(p), \cdots, f^m(p),\ h^1(q), \cdots, h^n(q)\right),$$

and

$$f \otimes h : M \times N \to E^{mn} : (p, q) \to (f(p), h(q)) = \left(f^1(p)h^1(q), \cdots, f^1(p)\ h^n(q), \cdots, f^m(p)h^n(q)\right).$$

Under certain conditions obtained by Decruyenaere and coauthors (1993), the tensor product map $f \otimes h$ is an immersion in the space E^{mn}.

The simplest examples of the tensor product immersions are tensor product surfaces. In the Euclidean space E^n, the tensor product surfaces of two

Euclidean plane curves, as well as of a Euclidean space curve and a Euclidean plane curve, are investigated by Mihai et al. (1995), Mihai & Rouxel (1995) and Arslan et al. (2001), respectively. Moreover, in the semi-Euclidean space E_ν^n, the tensor product surfaces of two Lorentzian plane curves, as well as of a Lorentzian plane curve and a Euclidean plane curve are studied by Mihai and coauthors (1995), and Mihai et al. (1995), respectively.

In this paper, we study tensor product surfaces of a Lorentzian space curve and a Euclidean plane curve and classify all minimal, totaly real and complex tensor product surfaces of such curves.

TENSOR PRODUCT SURFACES OF A LORENTZIAN SPACE CURVE AND A EUCLIDEAN PLANE CURVE

Let R_μ^m and R_ν^n be two pseudo-Euclidean spaces with metric matrices G_1 and G_2 respectively. We identify, in the usual way, the space R^{mn} with the space M of real $m \times n$ matrices. Let us consider the metric g in M given by:

$$g(A,\, B) = trace\big(G_1 A G_2 B^T\big),$$

where B^T denotes the transpose of B. Then (M,g) is isometric to the pseudo-Euclidean space $R_r^{mn}$of index $r = \mu(n-\nu) + \nu(m-\mu)$. The metric product $\otimes : R_\mu^m \times R_\nu^n \to$ M can be defined as $P \otimes Q = P^T Q$. Accordingly, the metric g in M is given by:

$$g(X \otimes V,\, Y \otimes W) = g_1(X, Y) g_2(V, W), \qquad (1)$$

where g_1 and g_2 are the metrics of R_μ^m and R_ν^n, respectively.

In particular, if $\alpha : R \to R_1^3$ and $\beta : R \to R^2$ are the Lorentzian space curve and a Euclidean plane curve respectively, then their tensor product $f(t,s) = \alpha(t) \otimes \beta(s)$ is defined as:

$$\otimes : R_1^3 \times R^2 \to R_2^6, \qquad f(t,s) = \alpha(t)^T \beta(s), \qquad (2)$$

where $\alpha(t)^T$ denotes the transpose of $\alpha(t)$. The pseudo-Riemannian metric g in R_2^6 is given by (1), where $g_1 = -dx_1^2 + dx_2^2 + dx_3^2$ and $g_2 = dx_1^2 + dx_2^2$ are the metrics of R_1^3 and R^2, respectively. By using equation (2), the canonical tangent vectors of $f(t,s)$ can be easily computed as

$$\frac{\partial f}{\partial t} = \alpha'(t)^T \beta(s) \text{ and } \frac{\partial \mathrm{f}}{\partial \mathrm{s}} = \alpha(\mathrm{t})^{\mathrm{T}} \beta'(\mathrm{s}) \qquad (3)$$

Hence relations (1), (2) and (3) imply that the coefficients of the pseudo-Riemannian metric, induced on $f(t,s)$ by the pseudo-Riemannian metric g of R_2^6, are:

$$g_{11} = g_1(\alpha', \alpha')g_2(\beta, \beta),$$

$$g_{12} = g_1(\alpha, \alpha')g_2(\beta, \beta')$$

$$g_{22} = g_1(\alpha, \alpha)g_2(\beta', \beta').$$

In the following, we will assume that α is a spacelike or a timelike curve with spacelike or timelike position vector and we will assume that β is a regular curve not passing through the origin. Hence $g_{11} \neq 0 \neq g_{22}$. We shall also assume that the tensor product surface $f(t,s)$ is a regular surface, i.e. $g_{11}g_{22} - g_{12}^2 \neq 0$. Consequently, an orthonormal basis for the tangent space is given by:

$$e_1 = \frac{1}{\sqrt{|g_{11}|}}\frac{\partial f}{\partial t} \text{ and}$$

$$e_2 = \frac{1}{\sqrt{\left|g_{11}\left(g_{11}g_{22} - g_{12}^2\right)\right|}}\left(g_{11}\frac{\partial f}{\partial s} - g_{12}\frac{\partial f}{\partial t}\right).$$

Recall that the mean curvature vector field $H$is defined by:

$$H = \frac{1}{2}(\varepsilon_1 h(e_1, e_1) + \varepsilon_2 h(e_2, e_2)),$$

where $h$is second fundamental form of $\alpha \otimes \beta$and $\varepsilon_i = g(e_i, e_i), \quad i = 1, 2$. In particular, by *Beltrami's* formula we have:

$$H = -\frac{1}{2}\Delta f.$$

Next, recall that a surface M in R_2^6 is said to be minimal if its mean curvature vector field H vanishes identically.

A basis of the normal space of $f(t,s)$ can be calculated as follows. Let $J_i : R_1^3 \to R_1^3$, $i = 1, 2$ and $J : R^2 \to R^2$ be the following maps

$$\begin{aligned} J_1(x,y,z) &= (y,x,0),\\ J_2(x,y,z) &= (0,z,-y), \\ J(x,y) &= (y,-x). \end{aligned} \tag{4}$$

Observe that $g_1 = (X, J_iX) = 0$ for $X \in R^3_1, i = 1,2$ and $g_2 = (Y, JY) = 0$ for $Y \in R^2$. Then a basis $\{n_1, n_2, n_3, n_4\}$ of the normal space is the following:

$$\begin{aligned} n_1(t,s) &= J_1\alpha(t) \otimes J\beta(s),\\ n_2(t,s) &= J_2\alpha(t) \otimes J\beta(s),\\ n_3(t,s) &= J_1\alpha'(t) \otimes J\beta'(s),\\ n_4(t,s) &= J_2\alpha'(t) \otimes J\beta'(s). \end{aligned} \tag{5}$$

Note that also the vector:

$$n_5(t,s) = J_3\alpha'(t) \otimes J\beta'(s), \tag{6}$$

is a normal vector, where $J_3 : R^3_1 \to R^3_1$ is the map defined by $J_3(x,y,z) = (z,0,x)$. It will be convenient later on to also consider this vector.

Accordingly, tensor product surface $f(t,s)$ is minimal in R^6_2 if and only if:

$$g(H,\, n_i) = 0, \quad i = 1,\, 2,\, 3,\, 4,\, 5.$$

On the other hand, by using *Beltrami's* formula, a surface $f(t,s)$ is minimal in R^6_2, if and only if:

$$g(\Delta f, n_i) = 0, \quad i = 1,\, 2,\, 3,\, 4,\, 5. \tag{7}$$

Since the Laplacian of $f(t,s)$ is given by

$$\begin{aligned} \Delta f &= g^{11}\frac{\partial^2 f}{\partial t^2} + 2g^{12}\frac{\partial^2 f}{\partial t \partial s} + g^{22}\frac{\partial^2 f}{\partial s^2} + \frac{1}{\sqrt{|\det(g_{ij})|}}\left[\frac{\partial}{\partial t}\left(\sqrt{|\det(g_{ij})|}g^{11}\right) + \frac{\partial}{\partial s}\left(\sqrt{|\det(g_{ij})|}g^{12}\right)\right]\frac{\partial f}{\partial t} \\ &+ \frac{1}{\sqrt{|\det(g_{ij})|}}\left[\frac{\partial}{\partial t}\left(\sqrt{|\det(g_{ij})|}g^{12}\right) + \frac{\partial}{\partial s}\left(\sqrt{|\det(g_{ij})|}g^{22}\right)\right]\frac{\partial f}{\partial s}, \end{aligned}$$

the minimality conditions (7) become

$$g\left(g_{11}\frac{\partial^2 f}{\partial s^2}-2g_{12}\frac{\partial^2 f}{\partial t\partial s}+g_{22}\frac{\partial^2 f}{\partial t^2},n_i\right)=0, \qquad i=1,\,2,\,3,\,4,\,5. \tag{8}$$

Recall that a *circle* in R^3_1 is defined as a planar curve with nonzero constant curvature.

In the first theorem, we classify all minimal tensor product surfaces in R^6_2.

Theorem 1. The tensor product immersion $f=\alpha\otimes\beta$ of a Lorentzian space curve $\alpha: R\to R^3_1$ and a Euclidean plane curve $\beta: R\to R^2$, is a minimal surface in R^6_2 if and only if:

(i) α is either the circle with the equation $\alpha(t)=r_0(\cosh(t),\,0,\,\sinh(t)),\ r_0\in R^+_0$ or hyperbolic spiral given by $\alpha(t)=a_1e^{a_2t}(\cosh(t),\,0,\,\sinh(t))$, $a_1\in R^+_0$, $a_2\in R_0$, $a_2\neq\pm1$, and β is orthogonal hyperbola with the equation $\beta(s)=\left(b/\sqrt{|\cos(2s)|}\right)(\sin(s),\cos(s))$, $b\in R_0$;

(ii) α is given by $\alpha(t)=\left(a/\sqrt{\cosh(2t)}\right)(\cosh(t),0,\sinh(t)),\ a\in R_0$, and β is either the circle with the equation $\beta(s)=\rho_0(\sin(s),\cos(s)),\ \rho_0\in R^+_0$ or logaritmic spiral given by $\beta(s)=b_1e^{b_2s}(\sin(s),\cos(s)),\ b_1\in R^+_0$, $b_2\in R_0$;

(iii) α is given either by

$$\alpha(t)=\frac{a_2}{\sqrt{\cosh((1-k)t+a_1)}}(\cosh(t),0,\sinh(t)),$$

or by

$$\alpha(t)=\frac{a_2}{\sqrt{|\sinh((1-k)t+a_1)|}}(\cosh(t),0,\sinh(t)),$$

and β is given by

$$\beta(s)=\frac{b_2}{\sqrt{|\cos((1+k)s+b_1)|}}(\sin(s),\cos(s)),$$

where, $a_1,\,b_1\in R,\ a_2,\,b_2,k\in R_0$ and $k\neq\pm1$;

(iv) α is the circle given by $\alpha(t)=\sqrt{c}(0,\,\cos(t/\sqrt{c}),\,\sin(t/\sqrt{c}))$, $c\in R^+_0$, and β is orthogonal hyperbola given by $\beta(s)=(b/\sqrt{|\cos(2s)|})\ (\sin(s),\,\cos(s))$, $b\in R_0$;

(v) α is the circle given by $\alpha(t)=\left(a/\sqrt{|\cos(2t)|}\right)(0,\,\sin(t),\,\cos(t)),\ a\in R_0$, and β is the circle with the equation $\beta(s)=\sqrt{c}(\,\cos(s/\sqrt{c}),\ \sin(s/\sqrt{c}))$, $c\in R^+_0$; and

(vi) α is given by $\alpha(t) = (a/\alpha_3(t),\ a/\alpha_3(t),\ \alpha_3(t))$, $a \in R_0^+, \alpha_3$ is arbitrary differentiable function, and β is the circle with the equation $\beta(s) = \sqrt{c}(\cos(s/\sqrt{c}),\ \sin(s/\sqrt{c}))$, $c \in R_0^+$.

Proof. Let $\alpha(t) = (\alpha_1(t),\ \alpha_2(t),\ \alpha_3(t))$ be a Lorentzian space curve and $\beta(s) = (\beta_1(s),\ \beta_2(s))$ be a Euclidean plane curve. Then their tensor product $f(t,s) = \alpha \otimes \beta$ is given by (2). Let us first suppose that $f(t,s)$ is a minimal surface in R_2^6. By using (2), we easily find:

$$\frac{\partial^2 f}{\partial t^2} = \alpha''(t) \otimes \beta(s), \quad \frac{\partial^2 f}{\partial s^2} = \alpha(t) \otimes \beta''(s) \text{ and } \frac{\partial^2 f}{\partial t \partial s} = \alpha'(t) \otimes \beta'(s). \tag{9}$$

The normal space of $f(t,s)$ is spanned by vectors $\{n_1, n_2, n_3, n_4\}$ given by (4). Consequently, relations (1), (5) and (9) yield:

$$g\left(\frac{\partial^2 f}{\partial t^2}, n_i\right) = g\left(\frac{\partial^2 f}{\partial s^2}, n_i\right) = g\left(\frac{\partial^2 f}{\partial t \partial s}, n_j\right) = 0, \quad i. = 1,2, \quad j = 3,4,5, \tag{10}$$

where n_5 is the normal vector given by (6). Next, relations, (1), (5), (6) and (9) imply:

$$\begin{aligned} g\left(\frac{\partial^2 f}{\partial t \partial s}, n_i\right) &= g_1(\alpha'(t),\ J_i\alpha(t))\, g_2(\beta'(s),\ J\beta(s)), \quad i = 1,2, \\ g\left(\frac{\partial^2 f}{\partial t^2}, n_j\right) &= g_1\big(\alpha''(t),\ J_{j-2}\alpha'(t)\big)\, g_2(\beta(s),\ J\beta'(s)), \quad j = 3,4,5, \\ g\left(\frac{\partial^2 f}{\partial s^2}, n_j\right) &= g_1\big(\alpha(t),\ J_{j-2}\alpha'(t)\big)\, g_2(\beta''(s),\ J\beta'(s)), \quad j = 3,4,5. \end{aligned} \tag{11}$$

Moreover, by using (8) and (10), we obtain that the minimality conditions are given by:

$$g\left(\frac{\partial^2 f}{\partial t \partial s}, n_i\right) = 0, \ g\left(g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2}, n_j\right) = 0, \quad i = 1,\ 2, \quad j = 3,4,5, \tag{12}$$

or else by:

$$g_{12} = 0, \quad g\left(g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2}, n_j\right) = 0, \quad j = 3,4,5. \tag{13}$$

Therefore, we may distinguish the following two cases.

Case I. Assume that conditions (12) hold. Then relations (11) and (12) imply:

$$\begin{aligned} & g_1(\alpha', J_i\alpha)g_2(\beta', J\beta) = 0, \quad i = 1,2, \\ & g_{22}g_1\left(\alpha'', J_{j-2}\alpha'\right)g_2(\beta, J\beta') + g_{11}g_1\left(\alpha, J_{j-2}\alpha'\right)g_2(\beta'', J\beta') = 0, \; j = 3,4,5. \end{aligned} \tag{14}$$

Next we consider two subcases.

Case I. 1. If $g_2(\beta', J\beta) = 0$, it follows that β is a straight line passing through the origin. Then $g_{11}g_{22} - g_{12}^2 = 0$, which means that the surface $f(t,s)$ is not regular. Hence we obtain a contradiction.

Case I. 2. If $g_1(\alpha', J_i\alpha) = 0$ for $i = 1,2$, the system of equation (14) becomes:

$$\begin{aligned} & -\alpha_1'\alpha_2 + \alpha_1\alpha_2' = 0, \\ & \alpha_2'\alpha_3 - \alpha_2\alpha_3' = 0, \\ & g_{22}\left(-\alpha_1''\alpha_2' + \alpha_1'\alpha_2''\right)g_2(\beta, J\beta') + g_{11}\left(-\alpha_1\alpha_2' + \alpha_1'\alpha_2\right)g_2(\beta'', J\beta') = 0, \\ & g_{22}\left(-\alpha_2'\alpha_3'' + \alpha_2''\alpha_3'\right)g_2(\beta, J\beta') + g_{11}\left(-\alpha_2'\alpha_3 + \alpha_2\alpha_3'\right)g_2(\beta'', J\beta') = 0, \\ & g_{22}\left(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3''\right)g_2(\beta, J\beta') + g_{11}\left(-\alpha_1\alpha_3' + \alpha_1'\alpha_3\right)g_2(\beta'', J\beta') = 0. \end{aligned} \tag{15}$$

Now we distinguish the following possibilities. If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, from the first two equation of (15) we get that α is a straight line passing through the origin. Then the surface $f(t,s)$ is not regular, which is a contradiction. Next if $\alpha_1 = 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, or $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 = 0$, in a similar way from (15) we obtain that α is a straight line passing through the origin, which implies a contradiction. Finally, if $\alpha_1 \neq 0$, $\alpha_2 = 0$ and $\alpha_3 \neq 0$, the system of equation (15) reduces to the equation:

$$g_{22}\left(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3''\right)g_2(\beta, J\beta') + g_{11}\left(-\alpha_1\alpha_3' + \alpha_1'\alpha_3\right)g_2(\beta'', J\beta') = 0.$$

Therefore, the last equation yields

$$\frac{g_1(\alpha,\alpha)\left(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3''\right)}{g_1(\alpha',\alpha')\left(\alpha_1\alpha_3' - \alpha_1'\alpha_3\right)} = \frac{g_2(\beta,\beta)g_2(\beta'', J\beta')}{g_2(\beta',\beta')g_2(\beta, J\beta')} = k, \quad k \in R_0. \tag{16}$$

Since α is arbitrary non-null curve with non-null position vector, lying in the timelike plane of R_1^3 with the equation $x_2 = 0$, we may assume that α has the equation $\alpha(t) = r(t)$ (cosh (t), 0, sinh (t)) or $\alpha(t) = r(t)$ (sinh (t), 0, cosh (t)). Without loss of generality, assume that:

$$\alpha(t) = r(t)(\cosh(t),\, 0,\, \sinh(t)). \tag{17}$$

Similarly, the equation of a regular curve β in R^2(which is not a straight line containing the origin) reads:

$$\beta(s) = \rho(s)(\sin(s),\ \cos(s)). \tag{18}$$

Equations (16), (17) and (18) imply:

$$\frac{\rho\rho'' - 2\rho'^2 - \rho^2}{\rho^2 + \rho'^2} = \frac{rr'' - 2r'^2 + r^2}{r^2 - r'^2} = k, \qquad k \in R_0. \tag{19}$$

If $k = 1$, from (19) we get two differential equations $\rho\rho'' - 3\rho'^2 - 2\rho^2 = 0$ and $rr'' - r'^2 = 0$, whose solutions are $\rho(s) = b_1/\sqrt{|\cos(2s + b_2)|}, b_1 \in R_0,\ b_2 \in R$, $r(t) = r_0 \in R_0^+$ or $r(t) = a_1 e^{a_2 t}$, $a_1 \in R_0^+$, $a_2 \in R_0$, $a_2 \neq \pm 1$. Taking $b_2 = 0$, we obtain that β is orthogonal hyperbola with the equation:

$$\beta(s) = \frac{b_1}{\sqrt{|\cos(2s)|}}(\sin(s), \cos(s)),$$

and α is the circle or hyperbolic spiral respectively given by:

$$\alpha(t) = r_0(\cosh(t),\, 0,\, \sinh(t)),$$

and

$$\alpha(t) = a_1 e^{a_2 t}(\cosh(t),\, 0,\, \sinh(t),$$

which proves statement (i).

If $k = -1$, in a similar way from equations (19) we get two differential equations $rr'' - 3r'^2 + 2r^2 = 0$ and $\rho\rho'' - \rho'^2 = 0$. It follows that $r(t) = a_3/\sqrt{\cosh(2t + a_4)}$, $a_3 \in R_0$, $a_4 \in R$, $\rho(s) = \ \rho_0 \in R_0^+$ or $\rho(s) = b_3 e^{b_4 s}$, $b_3 \in R_0^+$, $b_4 \in R_0$. Taking $a_4 = 0$, we obtain that α is the curve with the equation:

$$\alpha(t) = \frac{a_3}{\sqrt{\cosh(2t)}}(\cosh(t), 0, \sinh(t)),$$

and β is the circle or logaritmic spiral respectively given by

$$\beta(s) = \rho_0(\sin(s),\ \cos(s)),$$

and

$$\beta(s) = b_3 e^{b_4 s}(\sin(s),\ \cos(s)).$$

Hence statement (ii) is proved.

Finally, if $k \neq \pm 1$, putting $v = \rho'/\rho$ and $w = r'/r$ from (17) we find $v' = (1+k)(1+v^2)$ and $w' = (1-k)(w^2-1)$. After integration, we get $\rho(s) = b_1/\sqrt{|\cos((1 + k)s + b_2)|}$, $b_1 \in R_0$, $b_2 \in R$, $r(t) = a_3/\sqrt{\cosh((1-k)t + a_4)}$ if $|w| < 1$ or $r(t) = a_3/\sqrt{|\sinh((1-k)t + a_4)|}$ if $|w| > 1$, where $a_3 \in R_0$, $a_4 \in R$. Consequently, β is given by:

$$\beta(s) = \frac{b_1}{\sqrt{|\cos((1+k)s + b_2)|}}(\sin(s), \cos(s)),$$

and α is given either by:

$$\alpha(t) = \frac{a_3}{\sqrt{\cosh((1-k)t + a_4)}}(\cosh(t), 0, \sinh(t)),$$

or by:

$$\alpha(t) = \frac{a_3}{\sqrt{|\sinh((1-k)t + a_4)|}}(\cosh(t), 0, \sinh(t)).$$

In this way, statement (iii) is proved.

Case II. Assume that conditions (13) hold. Since $g_{12} = g_1(\alpha,\alpha')g_2(\beta,\beta') = 0$, we consider two subcases: $g_1(\alpha,\alpha') = 0$ and $g_2(\beta,\beta') = 0$.

Case II. 1. If $g_1(\alpha,\alpha') = 0$, then $g_1(\alpha,\alpha) = c,\ \ c \in R_0$, which means that α lies in pseudosphere S_1^2 if $c \in R_0^+$, or in pseudohyperbolical space H_0^2 if $c \in R_0^-$. By reparameterezing α, we may assume that $g_1(\alpha',\alpha') = \varepsilon = \pm 1$. Next, equations (11) and (13) imply:

$$g_{22}g_1(\alpha'', J_{j-2}\alpha')g_2(\beta, J\beta') + g_{11}g_1(\alpha, J_{j-2}\alpha')g_2(\beta'', J\beta') = 0, \quad j = 3,4,5, \quad (20)$$

and consequently,

$$\frac{c\left(-\alpha_1''\alpha_2'+\alpha_1'\alpha_2''\right)}{\varepsilon\left(\alpha_1\alpha_2'-\alpha_1'\alpha_2\right)}=\frac{c\left(\alpha_2''\alpha_3'-\alpha_2'\alpha_3''\right)}{\varepsilon\left(-\alpha_2\alpha_3'+\alpha_2'\alpha_3\right)}=$$
$$\frac{c\left(-\alpha_1''\alpha_3'+\alpha_1'\alpha_3''\right)}{\varepsilon\left(\alpha_1\alpha_3'-\alpha_1'\alpha_3\right)}=\frac{g_2(\beta,\beta)g_2\left(\beta'',J\beta'\right)}{g_2\left(\beta',\beta'\right)g_2\left(\beta,J\beta'\right)}=k, \quad k\in R_0. \tag{21}$$

From equation (21), we obtain the system of equations:

$$\begin{aligned}
c\left(-\alpha_1''\alpha_2'+\alpha_1'\alpha_2''\right)-k\varepsilon\left(\alpha_1\alpha_2'-\alpha_1'\alpha_2\right)&=0,\\
c\left(\alpha_2''\alpha_3'-\alpha_2'\alpha_3''\right)-k\varepsilon\left(-\alpha_2\alpha_3'+\alpha_2'\alpha_3\right)&=0,\\
c\left(-\alpha_1''\alpha_3'+\alpha_1'\alpha_3''\right)-k\varepsilon\left(\alpha_1\alpha_3'-\alpha_1'\alpha_3\right)&=0.
\end{aligned}$$

The above equations can be easily rewritten as:

$$\begin{aligned}
\left(c\alpha_2''+k\varepsilon\alpha_2\right)\alpha_1'-\alpha_2'\left(c\alpha_1''+k\varepsilon\alpha_1\right)&=0,\\
\left(c\alpha_2''+k\varepsilon\alpha_2\right)\alpha_3'-\alpha_2'\left(c\alpha_3''+k\varepsilon\alpha_3\right)&=0,\\
\left(c\alpha_3''+k\varepsilon\alpha_3\right)\alpha_1'-\alpha_3'\left(c\alpha_1''+k\varepsilon\alpha_1\right)&=0.
\end{aligned} \tag{22}$$

Note that the equations in (22) can be interpreted as the necessary and sufficient conditions for vectors $c\alpha''+k\varepsilon\alpha$ and α' to be linearly indepented. Hence there exist a function $\lambda(t)$ such that:

$$c\alpha''+k\varepsilon\alpha=\lambda\alpha'.$$

Taking the inner product on both sides of the previous equation with α' gives $\lambda(t)=0$. Therefore, $c\alpha''=-k\varepsilon\alpha$, which in particular implies that α lies in a plane π passing through the origin. If π is spacelike plane, up to isometries of R_1^3, we may assume that the normal to the plane is $(1, 0, 0)$. Then $\alpha_2^2+\alpha_3^2=c,\ \ c\in R_0^+$, which means that α is the circle given by

$$\alpha(t)=\sqrt{c}\left(0,\ \cos(t/\sqrt{c}),\ \sin(t/\sqrt{c})\right), \tag{23}$$

If π is timelike plane, up to isometries of R_1^3, we may assume that the normal to the plane is the vector $(0, 1, 0)$. Then $-\alpha_1^2+\alpha_3^2=c,\ \ c\in R_0$. Consequently, the curve α is the circle given by:

$$\alpha(t)=\sqrt{c}\left(\cosh(t/\sqrt{c}),\ 0,\ \sinh(t/\sqrt{c})\right),\ \ c\in R_0^+, \tag{24}$$

or by: $\alpha(t)=\sqrt{c}(\sinh(t/\sqrt{c}),\ 0, \cosh(t/\sqrt{c}),\ \ c\in R_0^+$.

If π is lightlike plane, up to isometries of R_1^3, we may assume that the normal to the plane is the vector $(1, 1, 0)$. Then $-\alpha_1^2+\alpha_1^2+\alpha_3^2=c$ and hence $\alpha_3(t)=\sqrt{c}$. Next the condition $g_1(\alpha',\alpha')=\pm1$ implies a contradiction.

Since β is a regular planar curve (which is not a straight line containing the origin), the equation of β is given by (18). If α is given by equation (23) (or by (24)), from (21) we easily obtain $k=1$. Then equations (18) and (22) imply differential equation $\rho\rho''-3\rho'^2-2\rho^2=0$. The solution of the previous equation is given by $\rho(s)=c_2/\sqrt{|\cos(2s+c_1)|},\quad c_1\in R,\quad c_2\in R_0$. We may take $c_1=0$, so β is orthogonal hyperbola with the equation:

$$\beta(s)=\frac{c_2}{\sqrt{|\cos(2s)|}}(\sin(s),\,\cos(s)),$$

which proves statement (iv).

Case II. 2. If $g_2(\beta,\beta')=0$, then $g_2(\beta,\beta)=c,\ c\in R_0^+$, which means that β is the circle with the equation $\beta(s)=\sqrt{c}(\cos(s/\sqrt{c}),\,\sin(s/\sqrt{c}))$. By using equation (18), we obtain:

$$\frac{g_1(\alpha,\alpha)\left(-\alpha_1''\alpha_2'+\alpha_1'\alpha_2''\right)}{g_1(\alpha',\alpha')\left(\alpha_1\alpha_2'-\alpha_1'\alpha_2\right)}=\frac{g_1(\alpha,\alpha)\left(\alpha_2''\alpha_3'-\alpha_2'\alpha_3''\right)}{g_1(\alpha',\alpha')\left(-\alpha_2\alpha_3'+\alpha_2'\alpha_3\right)}=\frac{g_1(\alpha,\alpha)\left(-\alpha_1''\alpha_3'+\alpha_1'\alpha_3''\right)}{g_1(\alpha',\alpha')\left(\alpha_1\alpha_3'-\alpha_1'\alpha_3\right)}=-1. \quad (25)$$

Putting $g_1(\alpha',\alpha')/g_1(\alpha,\alpha)=\lambda(t)$ in (25), we easily get the system of equations:

$$\begin{aligned}\left(-\alpha_1''+\lambda\alpha_1\right)\alpha_2'+\left(\alpha_2''-\lambda\alpha_2\right)\alpha_1'&=0,\\ \left(\alpha_2''-\lambda\alpha_2\right)\alpha_3'+\left(-\alpha_3''+\lambda\alpha_3\right)\alpha_2'&=0,\\ \left(-\alpha_1''+\lambda\alpha_2\right)\alpha_3'+\left(\alpha_3''-\lambda\alpha_3\right)\alpha_1'&=0.\end{aligned}$$

Consequently, there exist a function $m(t)$ such that $\alpha''-\lambda\alpha=m\alpha'$. It follows that α lies in the plane π passing through the origin. If π is spacelike plane, α is given by:

$$\alpha(t)=r(t)(0,\,\sin(t),\,\cos(t)). \quad (26)$$

Then equations (25) and (26) imply differential equation $rr''-3r'^2-2r^2=0$ with the solution $r(t)=b_2/\sqrt{|\cos(2t+b_1)|},\quad b_1\in R,\ b_2\in R_0$. Taking $b_1=0$, we get that α has the equation:

$$\alpha(t)=\frac{b_2}{\sqrt{|\cos(2t)|}}(0,\,\sin(t),\,\cos(t)).$$

This proves statement (v).

If π is timelike plane, the equation of α is reads:

$$\alpha(t) = r(t)(\cosh(t),\, 0,\, \sinh(t)). \tag{27}$$

By using equations (25) and (27), we find differential equation $rr'' - 3r'^2 + 2r^2 = 0$ and therefore $r(t) = a_2/\sqrt{\cosh(2t + a_1)}$, $a_1 \in R$, $a_2 \in R_0$. If we take $a_1 = 0$, it follows that α is given by:

$$\alpha(t) = \frac{a_2}{\sqrt{\cosh(2t)}}(\cosh(t),\, 0,\, \sinh(t)).$$

This case contained in statement (ii).

Finally, if π is lightlike plane in R^3_1 with the equation $x_1 = x_2$, the equation of α reads:

$$\alpha(t) = (\alpha_1(t),\, \alpha_1(t),\, \alpha_3(t)), \quad \alpha_3(t) \neq \text{constant}. \tag{28}$$

Equations (25) and (28) imply differential equation

$$\alpha_3^2(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3'') + \alpha_3'2(\alpha_1\alpha_3' - \alpha_1'\alpha_3) = 0.$$

From the last equation we get $\alpha_1(t) = c_0/\alpha_3(t)$, where $c_0 \in R_0^+$. Consequently, α is the curve given by:

$$\alpha(t) = (c_0/\alpha_3(t),\, c_0/\alpha_3(t),\, \alpha_3(t)),$$

where $\alpha_3(t)$is arbitrary differentiable function. In this way, statement (vi) is proved.

Conversely, if one of statements (i)-(vi) holds, a straightforward calculation shows that $f(t, s)$ is a minimal surface in R^6_2.

In the next two theorems, we classify totally real tensor product surfaces in the semi-Euclidean space R^6_2 and prove that there are no complex tensor product surfaces in the same space.

Theorem 2. The tensor product immersion $f(t, s) = \alpha(t) \otimes \beta(s)$ of a Lorentzian space curve $\alpha : R \to R^3_1$ and a Euclidean plane curve $\beta : R \to R^2$, is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure J_0 given by $J_0(p,\, q,\, u,\, v,\, z,\, w) = (-q,\, p, -v,\, u,\, -w,\, z)$ on R^6_2, if and only if α lies in a pseudosphere S^2_1or in a pseudohyperbolical space H^2_0.

Proof. Let $\alpha(t)$ be a Lorentzian space curve and $\beta(s)$ a Euclidean plane curve. Assume that the tensor product $f(t, s) = \alpha(t) \otimes \beta(s)$defined by (2) is a totally real Lorentzian immersion in R^6_2.Then there hold the conditions:

$$g\left(J_0\left(\frac{\partial f}{\partial t}\right), \frac{\partial f}{\partial s}\right) = 0, \quad g\left(J_0\left(\frac{\partial f}{\partial s}\right), \frac{\partial f}{\partial t}\right) = 0. \tag{29}$$

By using equations (2) and (3), we easily find:

$$J_0\left(\frac{\partial f}{\partial t}\right) = J_0(\alpha' \otimes \beta) = \alpha' \otimes J(-\beta), \quad J_0\left(\frac{\partial f}{\partial s}\right) = J_0(\alpha \otimes \beta') = \alpha \otimes J(-\beta'), \tag{30}$$

where J is the map defined by (4). Moreover, by using (1) and (30), conditions (29) become

$$\begin{aligned} g(\alpha' \otimes J(-\beta), \, \alpha \otimes \beta') = g_1(\alpha, \, \alpha')g_2(J(-\beta), \, \beta') = 0, \\ g(\alpha \otimes J(-\beta'), \, \alpha' \otimes \beta) = g_1(\alpha, \, \alpha')g_2(J(-\beta'), \beta) = 0. \end{aligned} \tag{31}$$

If $g_2(J(-\beta'), \, \beta) = -g_2(J(-\beta), \beta') = 0$, it follows that β is a straight line passing through the origin, which implies that a surface $f(t, s)$ is not regular. Hence we obtain a contradiction.

On the other hand, if $g_1(\alpha, \, \alpha') = 0$, then $g_1(\alpha, \, \alpha) = c, \;\; c \in R_0$,which means that α lies in a pseudosphere S_1^2 if $c > 0$, or in a pseudohyperbolical space H_0^2 if $c < 0$. Conversely, if α lies in a pseudosphere or in a pseudohyperbolical space, then equations (1), (2), (3) and (27) imply that the conditions (25) are satisfied. Therefore, $f(t, s)$ is a totally real surface which proves the theorem.

Theorem 3. There are no complex tensor product surfaces $f(t, s) = \alpha(t) \otimes \beta(s)$ of a Lorentzian space curve $\alpha : R \to R_1^3$ and a Euclidean plane curve $\beta : R \to R^2$, with respect to the pseudo-Hermitian structure J_0 given by $J_0(p, \, q, \, u, \, v, \, z, \, w) = (-q, \, p, -v, \, u, \, -w, \, z)$ on R_2^6.

Proof. Let us suppose that the tensor product $f(t, s)$ of a Lorentzian space curve $\alpha(t)$ and a Euclidean plane curve $\beta(s)$defined by (2) is a complex Lorentzian immersion. By definition, the following equations are satisfied:

$$g\left(J_0\left(\frac{\partial f}{\partial t}\right), \, n_i\right) = 0, \quad g\left(J_0\left(\frac{\partial f}{\partial s}\right), \, n_i\right) = 0, \quad i = 1, \, 2, \, 3, \, 4, \, 5, \tag{32}$$

where $\{n_1, n_2, n_3, n_4\}$ given by (4) is a basis of the normal space and $n_5 \in span\{n_1, n_2, n_3, n_4\}$ is a normal vector given by (6). On the other hand, equations (1), (5), (6) and (30) imply:

$$g\left(J_0\left(\frac{\partial f}{\partial s}\right), n_i\right) = 0, \; i = 1, 2, \quad g\left(J_0\left(\frac{\partial f}{\partial t}\right), n_j\right) = 0, \; j = 3, 4, 5,$$
$$g\left(J_0\left(\frac{\partial f}{\partial t}\right), n_i\right) = -g_1(\alpha', J_i\alpha)g_2(J\beta, J\beta), \; i = 1, 2, \tag{30}$$
$$g\left(J_0\left(\frac{\partial f}{\partial s}\right), n_j\right) = -g_1(\alpha, J_{j-2}\alpha')g_2(J\beta', J\beta'), \; j = 3, 4, 5.$$

Since $g_1(\alpha, J_i\alpha') = -g_1(\alpha', J_i\alpha), \; i = 1, 2, 3$, from equations (32) and (33) we obtain:

$$g_1(\alpha, J_{j-2}\alpha') = 0, \; j = 3, 4, 5,$$

which is equivalent with

$$\alpha_1'\alpha_2 - \alpha_1\alpha_2' = 0, \; \alpha_2'\alpha_3 - \alpha_2\alpha_3' = 0, \; \alpha_1'\alpha_3 - \alpha_1\alpha_3' = 0.$$

It follows that α is a straight line passing through the origin, but then $f(t, s)$ is not regular surface, which gives a contradiction.

ACKNOWLEDGEMENTS

The authors are very grateful to the referees for their useful comments and suggestions which improved the first version of the paper.

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Submitted : *28/9/2005*
Revised : *21/3/2007*
Accepted : *1/4/2007*

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خلاصة

هذا البحث يصنف ناتج موتر السطوح للوراينتزان الصفري، الكلي، الحقيقي والمركب والمنحنى المستو الإقليدي.