

Random fixed point of Greguš mapping and its application to nonlinear stochastic integral equations

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ABSTRACT

We obtain sufficient conditions for the existence of random fixed point of Greguš type random operators on separable Banach spaces and use it to solve a random nonlinear integral equation of the form:

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s).$$

To further illustrate, examples of nonlinear stochastic integral equation are constructed.

Keywords: Random fixed point; Greguš mapping; nonlinear stochastic integral equation; banach space.

INTRODUCTION

In recent years, considerable attention has been given to the study of random operators involving probabilistic models with numerous applications. The major area concerning random operator equation involves the existence, uniqueness, characterization, contraction and approximation of solutions. The introduction of randomness however leads to several new equations of measurability of solutions.

It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems. Hanš (1961) first formulated a random analogue of the classical Banach contraction principle in a separable

Banach space. Subsequently fundamental work has been done in this area by Bharucha-Reid (1972); Itoh (1979); Mukherjee (1966) and Sehgal & Waters (1984); Papageorgiou (1986); Beg (1977, 2002a, 2002b); Beg & Shahzad (1994); Xu & Beg (1998); Saha & Debnath (2007); Saha (2006) and Tan & Yuan (1997). Still there are some new questions and problems that expand the scope of the random fixed point theory. Itoh (1979) and Padgett (1973) applied random fixed point theorem to prove the existence of a solution in a Banach space of a random nonlinear integral equation. Achari (1983), Saha & Dey (2012) developed this new area of application. Recently Saha & Ganguly (2012) extended this application for Ćirić (1974) operator which is generalization of Greguš (1980) mapping. The random fixed point theorems for Greguš (1980) mapping is not worked out by the researchers. So we continue to investigate this problem as an application in the light of the random analogue of certain class of mappings which is more general than semi non-expansive mappings on a separable Banach space. We consider here a class of contractive operator due to Greguš (1980) and have been able to prove random analogue of deterministic fixed point theorems for such operators belonging to this class. As an application, we obtained the existence of a solution in a Banach space of a random nonlinear integral equation of the form:

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s)$$

where

- (i) S is a locally compact metric space with a metric d on $S \times S$ equipped with a complete σ -finite measure μ_0 defined on the collection of Borel subsets of S ;
- (ii) $\omega \in \Omega$, where ω is a supporting element of a set of probability measure space (Ω, β, μ) ;
- (iii) $x(t; \omega)$ is the unknown vector-valued random variable for each $t \in S$;
- (iv) $h(t; \omega)$ is the stochastic free term defined for $t \in S$;
- (v) $k(t, s; \omega)$ is the stochastic kernel defined for t and s in S and
- (vi) $f(t, x)$ is a vector-valued function of $t \in S$ and x .

PRELIMINARIES

In order to make the paper self contained, first we state some important definitions and an example that are available in Joshi & Bose (1984) and Debnath & Mikusinski (2005).

Let (X, β_X) be a separable Banach space where β_X is a σ -algebra of Borel subsets of X , and let (Ω, β, μ) denote a complete probability measure space with measure μ , and β be a σ -algebra of subsets of Ω . For more details one can see Joshi & Bose (1984).

Definition 2.1. A mapping $x : \Omega \rightarrow X$ is said to be an X -valued random variable, if the inverse image under the mapping x of every Borel set B of X belongs to β , that is $x^{-1}(B) \in \beta$, for all $B \in \beta_X$.

Definition 2.2. A mapping $x : \Omega \rightarrow X$ is said to be a finitely valued random variable, if it is constant on each of finite number of disjoint sets $A_i \in \beta$ and is equal to 0 on $\Omega - \left(\bigcup_{i=1}^n A_i\right)$. x is called a simple random variable, if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

Definition 2.3. A mapping $x : \Omega \rightarrow X$ is said to be a strong random variable, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables, which converges to $x(\omega)$ almost surely, that is, there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega), \omega \in \Omega - A_0$.

Definition 2.4. A mapping $x : \Omega \rightarrow X$ is said to be a weak random variable, if the function $x^*(x(\omega))$ is a real valued random variables for each $x^* \in X^*$, the space X^* denoting the first normed dual space of X .

In a separable Banach space X , the notions of strong and weak random variables $x : \Omega \rightarrow X$ (see corollary 1 of Joshi & Bose (1984)) coincide and in respect of such a space X , x is termed as a random variable.

We recall the following results with appropriate references.

Theorem 2.5. Joshi & Bose (1984, Theorem 6.1.2(a))

Let $x, y : \Omega \rightarrow X$ be strong random variables and α, β be constants. Then the following statements hold.

- (a) $\alpha x(\omega) + \beta y(\omega)$ is a strong random variable,
- (b) If $f(\omega)$ is a real valued random variable and $x(\omega)$ is a strong random variable, then $f(\omega)x(\omega)$ is a strong random variable.
- (c) If $\{x_n(\omega)\}$ is a sequence of strong random variables converging strongly to $x(\omega)$ almost surely, i.e., if there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n \rightarrow \infty} \|x_n(\omega) - x(\omega)\| = 0$ for every $\omega \notin A_0$, then $x(\omega)$ is a strong random variable.

Remark 2.6. If X is a separable Banach space, then every strong and also weak random variable is measurable in the sense of definition 2.1.

Let Y be another Banach space. We also need the following definitions from Joshi & Bose (1984).

Definition 2.7. A mapping $F: \Omega \times X \rightarrow Y$ is said to be a random mapping if $F(\omega, x) = Y(\omega)$ is a Y -valued random variable for every $x \in X$.

Definition 2.8. A mapping $F: \Omega \times X \rightarrow Y$ is said to be a continuous random mapping, if the set of all $\omega \in \Omega$ for which $F(\omega, x)$ is a continuous function of x has measure one.

Definition 2.9. A mapping $F: \Omega \times X \rightarrow Y$ is said to be demi-continuous at the $x \in X$ if $\|x_n - x\| \rightarrow 0$ implies $F(\omega, x_n) \text{ weakly } F(\omega, x)$ almost surely.

Theorem 2.10. Joshi & Bose (1984) Theorem 6.2.2) Let $F: \Omega \times X \rightarrow Y$ be a demi-continuous random mapping where Banach space Y is separable. Then for any X -valued random variable x , the function $F(\omega, x(\omega))$ is a Y -valued random variable.

Remark 2.11. Joshi & Bose (1984) Since a continuous random mapping is a demi-continuous random mapping, Theorem 2.5 is also true for a continuous random mapping.

We shall also recall the following definitions as seen in Joshi & Bose (1984).

Definition 2.12. An equation of the type $F(\omega, x(\omega)) = x(\omega)$ where $F: \Omega \times X \rightarrow X$ is a random mapping is called a random fixed point equation.

Definition 2.13. Any mapping $x: \Omega \rightarrow X$ which satisfies random fixed point equation $F(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.

Definition 2.14. Any X -valued random variable $x(\omega)$ which satisfies $\mu\{\omega: F(\omega, x(\omega)) = x(\omega)\} = 1$ is said to be a random solution of the fixed point equation or a random fixed point of F .

Remark 2.15. A random solution is a wide sense solution of the fixed point equation. But the converse is not necessarily true. This is evident from the following example as found under Remark 1 in Joshi & Bose (1984).

Example 2.16. Let X be the set of all real numbers and let E be a non measurable subset of X . Let $F: \Omega \times X \rightarrow X$ be random mapping defined as $F(\omega, x) = x^2 + x - 1$ for all $\omega \in \Omega$.

In this case the real valued function $x(\omega)$, defined as $x(\omega) = 1$ for all $\omega \in \Omega$ is a random fixed point of F . However the real valued function $y(\omega)$ defined as

$$y(\omega) = \begin{cases} -1, & \omega \notin E \\ 1, & \omega \in E \end{cases}$$

is a wide sense solution of the fixed point equation $F(\omega, x(\omega)) = x(\omega)$, without being a random fixed point of F .

Before analyzing the main result, it would be better to state the deterministic fixed point theorem of Greguš (1980).

Theorem 2.17. Greguš (1980) Let X be a Banach space and C a closed convex subset of X . Let $T : C \rightarrow C$ be a mapping satisfying

$$\|Tx - Ty\| \leq a\|x - y\| + p\|Tx - x\| + p\|Ty - y\|$$

for all $x, y \in C$, where $0 < a < 1, p \geq 0$ and $a + 2p = 1$. Then T has a unique fixed point.

We now prove the random analogue of Greguš fixed point theorem.

RANDOM FIXED POINT

Theorem 3.1. Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let

$T : \Omega \times X \rightarrow X$ be a continuous random operator such that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq a(\omega)\|x_1 - x_2\| + p(\omega)\|T(\omega, x_1) - x_1\| + p(\omega)\|T(\omega, x_2) - x_2\|$$

almost surely for all $x_1, x_2 \in X$, where $a(\omega)$ and $p(\omega)$, $\omega \in \Omega$ are real valued random variables such that $0 < a(\omega) < 1, p(\omega) \geq 0$ satisfying $a(\omega) + 2p(\omega) < 1$ almost surely then there exists a unique random fixed point of T .

Proof. Let $A = \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\}$

$$C_{x_1, x_2} = \{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \leq a(\omega)\|x_1 - x_2\| + p(\omega)\|T(\omega, x_1) - x_1\| + p(\omega)\|T(\omega, x_2) - x_2\|\}$$

and $B = \{\omega \in \Omega : a(\omega) + 2p(\omega) = 1\} \cap \{\omega \in \Omega : 0 < a(\omega) < 1, p(\omega) \geq 0\}$.

Let S be a countable dense subset of X .

We will show that $\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) = \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)$

Let $\omega \in \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)$ then for all $s_1, s_2 \in S$,

$$\|T(\omega, s_1) - T(\omega, s_2)\| \leq a(\omega)\|s_1 - s_2\| + p(\omega)\|T(\omega, s_1) - s_1\| + p(\omega)\|T(\omega, s_2) - s_2\| \quad (3.1)$$

Let $x_1, x_2 \in X$, we have,

$$\begin{aligned}
\|T(\omega, x_1) - T(\omega, x_2)\| &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_1) - T(\omega, s_2)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\
&\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\
&\quad + a(\omega)\|s_1 - s_2\| + p(\omega)\|T(\omega, s_1) - s_1\| + p(\omega)\|T(\omega, s_2) - s_2\| \\
&\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\
&\quad + a(\omega)(\|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\|) \\
&\quad + p(\omega)(\|s_1 - x_1\| + \|x_1 - T(\omega, x_1)\| + \|T(\omega, x_1) - T(\omega, s_1)\|) \\
&\quad + p(\omega)(\|s_2 - x_2\| + \|x_2 - T(\omega, x_2)\| + \|T(\omega, x_2) - T(\omega, s_2)\|) \\
&\leq (1 + p(\omega))\|T(\omega, x_1) - T(\omega, s_1)\| + (1 + p(\omega))\|T(\omega, s_2) - T(\omega, x_2)\| \\
&\quad + (a(\omega) + p(\omega))\|s_1 - x_1\| + (a(\omega) + p(\omega))\|x_2 - s_2\| \\
&\quad + a(\omega)\|x_1 - x_2\| + p(\omega)\|x_1 - T(\omega, x_1)\| + p(\omega)\|T(\omega, x_2) - x_2\| \tag{3.2}
\end{aligned}$$

Since for a particular $\omega \in \Omega$, $T(\omega, x)$ is a continuous function of x , so for any $\varepsilon > 0$, there exists $\delta_i(x_i) > 0$; ($i = 1, 2$) such that $\|T(\omega, x_1) - T(\omega, s_1)\| < \varepsilon/4$ whenever $\|x_1 - s_1\| < \delta_1(x_1)$ and $\|T(\omega, x_2) - T(\omega, s_2)\| < \frac{\varepsilon}{4}$ whenever $\|x_2 - s_2\| < \delta_2(x_2)$.

Now we choose $\delta_1 = \min\left(\delta_1(x_1), \frac{\varepsilon}{4}\right)$, $\delta_2 = \min\left(\delta_2(x_2), \frac{\varepsilon}{4}\right)$. For such a choice of δ_1, δ_2 we get from (3.2), $\|T(\omega, x_1) - T(\omega, x_2)\| \leq (1 + p(\omega))\frac{\varepsilon}{4} + (1 + p(\omega))\frac{\varepsilon}{4}$

$$\begin{aligned}
&\quad + (a(\omega) + p(\omega))\varepsilon/4 + (a(\omega) + p(\omega))\frac{\varepsilon}{4} \\
&\quad + a(\omega)\|x_1 - x_2\| + p(\omega)\|T(\omega, x_1) - x_1\| + p(\omega)\|T(\omega, x_2) - x_2\| \\
&\leq \varepsilon + a(\omega)\|x_1 - x_2\| + p(\omega)\|T(\omega, x_1) - x_1\| \\
&\quad + p(\omega)\|T(\omega, x_2) - x_2\|
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq a(\omega)\|x_1 - x_2\| + p(\omega)\|T(\omega, x_1) - x_1\| + p(\omega)\|T(\omega, x_2) - x_2\|$$

Therefore $\omega \in \bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B$ which implies $\bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B \subset \bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B$.
Again $\bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B \subset \bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B$

$$\text{So } \bigcap_{x_1, x_2 \in X} C_{x_1, x_2} \cap A \cap B = \bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B.$$

$$\text{Let } N' = \bigcap_{s_1, s_2 \in S} C_{s_1, s_2} \cap A \cap B,$$

$$\text{Then } \mu(N') = 1$$

So, for each $\omega \in N'$, $T(\omega, x)$ is a deterministic operator satisfying Greguš (10). Hence T has a unique random fixed point in X .

APPLICATION TO A RANDOM NONLINEAR INTEGRAL EQUATION

Here we apply Theorem 3.1 to prove the existence of a solution in a Banach space of a random nonlinear integral equation of the form:

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s) \tag{4.1}$$

where

- (vvi) S is a locally compact metric space with a metric d on $S \times S$ equipped with a complete σ -finite measure μ_0 defined on the collection of Borel subsets of S ;
- (viii) $\omega \in \Omega$, where ω is a supporting element of a set of probability measure space (Ω, β, μ) ;
- (ix) $x(t; \omega)$ is the unknown vector-valued random variable for each $t \in S$;
- (x) $h(t; \omega)$ is the stochastic free term defined for $t \in S$;
- (xi) $k(t, s; \omega)$ is the stochastic kernel defined for t and s in S and
- (xii) $f(t, x)$ is a vector-valued function of $t \in S$ and x .

The integral in equation (4.1) is interpreted as a Bochner integral Padgett (1973).

We shall further assume that S is the union of a decreasing sequence of countable family of compact sets $\{C_n\}$, such that for any other compact set in S there is a C_i , which contains it Arens (1946).

Definition 4.1. We define the space $C(S, L_2(\Omega, \beta, \mu))$ to be the space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compact sets of S that is for each fixed $t \in S$, $x(t; \omega)$ is a vector valued random variable such that

$$\|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}^2 = \int_{\Omega} |x(t; \omega)|^2 d\mu(\omega) < \infty$$

It may be noted that $C(S, L_2(\Omega, \beta, \mu))$ is locally convex space, whose topology is defined by a countable family of semi-norms given by

$$\|x(t; \omega)\|_n = \sup_{t \in C_n} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}, n = 1, 2, \dots \quad \text{Yosida(1965)}$$

Moreover, $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology, since $L_2(\Omega, \beta, \mu)$ is complete.

We further define $BC = BC(S, L_2(\Omega, \beta, \mu))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ with norm

$$\|x(t; \omega)\|_{BC} = \sup_{t \in S} \|x(t; \omega)\|_{L_2(\Omega, \beta, \mu)}$$

Here the space $BC \subset C$ is the space of all second order vector valued stochastic process defined on S , which is bounded and continuous in mean square. We will consider the function $h(t; \omega)$ and $f(t, x(t; \omega))$ to be in the space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We assume that for each pair $(t, s), k(t, s; \omega) \in L_\infty(\Omega, \beta, \mu)$ and denote the norm by

$$\begin{aligned} \| \|k(t, s; \omega)\| \| &= \|k(t, s; \omega)\|_{L_\infty(\Omega, \beta, \mu)} \\ &= \mu - \text{ess sup}_{\omega \in \Omega} |k(t, s; \omega)| \end{aligned}$$

Let us suppose that $k(t, s; \omega)$ is such that $\| \|k(t, s; \omega)\| \| \cdot \|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$ is μ_0 -integrable with respect to s for each $t \in S$ and $x(s; \omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ and there exists a real valued function G defined $\mu_0 - a.e.$ on S , so that $G(S) \|x(s; \omega)\|_{L_2(\Omega, \beta, \mu)}$ is μ_0 -integrable and for each pair $(t, s) \in S \times S$,

$$\| \|k(t, u; \omega) - k(s, u; \omega)\| \| \cdot \|x(u, \omega)\|_{L_2(\Omega, \beta, \mu)} \leq G(u) \|x(u, \omega)\|_{L_2(\Omega, \beta, \mu)}$$

$\mu_0 - a.e.$ Further, for almost all $s \in S, k(t, s; \omega)$ will be continuous in t from S into $L_\infty(\Omega, \beta, \mu)$.

We now define the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega)d\mu_0(s) \tag{4.2}$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S, (Tx)(t; \omega) \in L_2(\Omega, \beta, \mu)$ and that $(Tx)(t; \omega)$ is continuous in mean square by Lebesgue dominated convergence theorem. So $(Tx)(t; \omega) \in C(S, L_2(\Omega, \beta, \mu))$

Definition 4.2. Lee & Padgett (1977) Let B and D be two Banach spaces. The pair (B, D) is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.

Lemma 4.3. Joshi & Bose (1984) The linear operator T defined by (4.2) is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.

Lemma 4.4. Joshi & Bose (1984) and Lee & Padgett (1977) If T is a continuous linear operator from $C(S, L_2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to T , then T is continuous from B into D .

Remark 4.5. Joshi & Bose (1984) The operator T defined by (4.2) is a bounded linear operator from B into D .

A random solution of the equation (4.1) will mean a function $x(t; \omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ which satisfies the equation (4.1) $\mu - a.e.$

We are now in a position to prove the following theorem.

Theorem 4.6. We consider the stochastic integral equation (4.1) subject to the following conditions

- (a) B and D are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to the integral operator defined by (4.2);
- (b) $x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set

$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$ into the space B satisfying

$$\begin{aligned} & \|f(t, x_1(t; \omega)) - f(t, x_2(t; \omega))\|_B \leq a(\omega) \|x_1(t; \omega) - x_2(t; \omega)\|_D \\ & + p(\omega) [\|x_1(t; \omega) - f(t, x_1(t; \omega))\|_D + \|x_2(t; \omega) - f(t, x_2(t; \omega))\|_D] \end{aligned} \quad (4.3)$$

for $x_1(t; \omega), x_2(t; \omega) \in Q(\rho)$, where $0 < a(\omega) < 1$ and $p(\omega) \geq 0$ are real valued random variable satisfying $a(\omega) + 2p(\omega) = 1$ almost surely.

- (c) $h(t; \omega) \in D$

Then there exists a unique random solution of (4.1) in $Q(\rho)$, provided $\frac{c(\omega)}{1 - p(\omega)} < 1$ and $\|h(t; \omega)\|_D + (1 + c(\omega))\|f(t; 0)\|_B \leq \rho(1 - c(\omega))$ where $c(\omega)$ is the norm of $T(\omega)$.

Proof. Define the operator $U(\omega)$ from $Q(\rho)$ into D by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega)) d\mu_0(s)$$

Now $\|(Ux)(t; \omega)\|_D \leq \|h(t; \omega)\|_D + c(\omega)\|f(t, x(t; \omega))\|_B$

$$\leq \|h(t; \omega)\|_D + c(\omega)\|f(t; 0)\|_B + c(\omega)\|f(t, x(t; \omega)) - f(t; 0)\|_B$$

Then from the condition (4.3) of this theorem

$$\begin{aligned} \|f(t, x(t; \omega)) - f(t; 0)\|_B &\leq a(\omega)\|x(t; \omega)\|_D + p(\omega)[\|x(t; \omega) - f(t, x(t; \omega))\|_D \\ &\quad + p(\omega)\|f(t; 0)\|_D] \\ &\leq a(\omega)\|x(t; \omega)\|_D + p(\omega)\|x(t; \omega)\|_D \\ &\quad + p(\omega)\|f(t, x(t; \omega)) - f(t; 0)\|_D + 2p(\omega)\|f(t; 0)\|_D \end{aligned}$$

Hence $\|f(t, x(t; \omega) - f(t; 0))\|_B \leq \frac{a(\omega) + p(\omega)}{1 - p(\omega)} \rho + \frac{2p(\omega)}{1 - p(\omega)} \|f(t; 0)\|_B$

$$= \rho + 2p(\omega)1 - p(\omega)\|f(t; 0)\|_B \tag{4.4}$$

Therefore by (4.4)

$$\begin{aligned} \|(Ux)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + c(\omega)\|f(t; 0)\|_B + c(\omega) \left[\rho + \frac{2p(\omega)}{1 - p(\omega)} \|f(t; 0)\|_B \right] \\ &< \|h(t; \omega)\|_D + c(\omega)\|f(t; 0)\|_B + c(\omega)\rho + \frac{c(\omega)}{1 - p(\omega)} \|f(t; 0)\|_B \\ &< \|h(t; \omega)\|_D + (1 + c(\omega))\|f(t; 0)\|_B + c(\omega)\rho \\ &< \rho \end{aligned}$$

Hence $(Ux)(t; \omega) \in Q(\rho)$. Then for $x_1(t; \omega), x_2(t; \omega) \in Q(\rho)$, we have by condition (b)

$$\begin{aligned} &\|(Ux_1)(t; \omega) - (Ux_2)(t; \omega)\|_D \\ &= \left\| \int_S k(t, s; \omega)[f(s, x_1(s; \omega)) - f(s, x_2(s; \omega))]d\mu_0(s) \right\|_D \\ &\leq c(\omega)\|f(t, x_1(t; \omega)) - f(t, x_2(t; \omega))\|_B \\ &\leq a(\omega)\|x_1(t; \omega) - x_2(t; \omega)\|_D \\ &\quad + p(\omega)[\|x_1(t; \omega) - (Ux_1)(t; \omega)\|_D + \|x_2(t; \omega) - (Ux_2)(t; \omega)\|_D] \end{aligned}$$

Since $\frac{c(\omega)}{1 - p(\omega)} < 1$. Thus $U(\omega)$ is a contractive nonlinear operator on $Q(\rho)$. Hence by Theorem 3.1 there exists a random fixed point $x(t, \omega)$ of $U(\omega)$, which is the random solution of the equation (4.1).

Example 4.7. Consider the following nonlinear stochastic integral

$$x(t; \omega) = \int_0^\infty \frac{e^{-t-s}}{8(1 + |x(s; \omega)|)} ds$$

Comparing with (4.1) we see that

$$h(t; \omega) = 0, k(t, s; \omega) = \frac{1}{4}e^{-t-s}, f(s, x(s; \omega)) = \frac{1}{2(1 + |x(s; \omega)|)}$$

By routine calculation, it is easy to show that (4.3) is satisfied with $a(\omega) = \frac{1}{2}$ and $0 \leq p(\omega) \leq \frac{1}{4}$.

Comparing with integral operator equation (4.2), we see that of norm of $T(\omega)$ is $c(\omega) = \frac{1}{8}$.

Also we see that $\frac{c(\omega)}{1 - p(\omega)} < 1$. So all the conditions of Theorem 4.6 are satisfied and hence there exists a random fixed point of the integral operator T satisfying (4.2).

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نقطة ثابتة عشوائية لتطبيق غريغوش وتطبيقاتها على معادلات تكاملية تصادفية غير خطية

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خلاصة

نحصل في هذا البحث على شروط كافية لوجود نقطة ثابتة عشوائية لمؤثرات عشوائية من نوع غريغوش على فضاءات بناخ القابلة للفصل. نقوم بعد ذلك باستخدام نتائجننا لحل معادلات تفاضلية عشوائية غير خطية. ولإيضاح نتائجننا نقوم بإنشاء أمثلة لمعادلات تكاملية تصادفية غير خطية.