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Some applications of differential subordination on certain class of analytic functions defined by integral operator

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ABSTRACT

Two-parameters function $H(n, \lambda; z)$ involving the Feltt multiplier operator is introduced. Subordination properties as well as sufficient conditions for starlikeness are also obtained.

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INTRODUCTION

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in C : \ |z| < 1\}$. Let S, $S^*(\alpha)$ and $C(\alpha)$ $(0 \leq \alpha < 1)$ be the subclasses of functions in A which are, respectively, univalent, starlike of order α and convex of order α in U. Denote by $S^*(0) = S^*$ and $C(0) = C$. Suppose also that P denotes the class of functions $k(z)$ given by

$$k(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{1.2}$$

which are analytic in $U$and satisfy the inequality

$$Re(k(z)) > 0 \quad (z \in U).$$

If f and g are analytic in U , we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)), \ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (Miller & Mocanu, 2000; Bulboacă, 2005):

$f(z) \prec g(z) \quad \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

For a function $f(z)$ given by (1.1) and $g(z)$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.3}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.4}$$

For an analytic function $f(z)$ given by (1.1) and for $n \in N_0$, Flett (1972) defined the multiplier transformations $I^n f$ by

$$I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \qquad (z \in U;\ n \in N_0). \tag{1.5}$$

Clearly, the function $I^n f(z)$ is analytic in U . We note that for $n \in N_0$, we have

(i) $I^{-n} f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k = D^n f(z)$,

and

(ii) $\quad z(I^{-n} f(z))' = D^{n+1} f(z)$,

where the operator $D^n f$ was introduced by Sălăgean (1983). We also note that

$I^n(I^m f(z)) = I^{n+m} f(z) \qquad (z \in U)$

for all integers n and m. Further, the operator I^n can be seen as a convolution of two functions. That is

$I^n f(z) = (h * h * \ldots * h * f)(z)$,

where the function $h(z) = \log \dfrac{1}{1-z} = z + \sum_{k=2}^{\infty} k^{-1} z^k$ occurs n times. It follows from (1.5) that

$$z(I^n f(z))' = I^{n-1} f(z) \qquad (n \in Z) \tag{1.6}$$

and

$$I^0 f(z) = f(z), \quad I^{-1} f(z) = z f'(z) \quad , \quad I^{-2} f(z) = z(f'(z) + z f''(z)).$$

We now define a two-parameters function $H(n, \lambda; z)$ by

$$H(n, \lambda; z) = (1-\lambda)\frac{I^{n-1}f(z)}{I^{n}f(z)} + \lambda\frac{I^{n-2}f(z)}{I^{n-1}f(z)} \quad (z \in U;\ \lambda \in R;\ n \in Z; f \in A). \tag{1.7}$$

Finally, we denote by $K(n, \lambda, \alpha)$ the class of functions $f(z) \in A$, which satisfy the following condition:

$Re(H(n, \lambda; z)) > \alpha \quad (z \in U;\ 0 \le \alpha < 1;\ \lambda \in R;\ n \in Z)$.

We note that :

(i) $K(0, \lambda, \alpha) = M(\lambda, \alpha)$ $(\lambda \ge 0;\ 0 \le \alpha < 1)$, is the class of $\lambda-$ convex functions of order α (Srivastava & Attiya, 2007);

(ii) $K(0, \lambda, 0) = M(\lambda)(\lambda \ge 0)$, is the class of $\lambda-$ convex functions (Miller *et al.*,1973; Mocanu, 1969; Mocanu, 1994):

(iii) $K(0, 0, \alpha) = S^*(\alpha)$ and $K(0, 1, \alpha) = C(\alpha)$ (Srivastava & Owa, 1992).

Consider the first-order differential subordination

$$H(\varphi(z), z\varphi'(z); z) \prec h(z).$$

A univalent function q is called its dominant if $\varphi(z) \prec q(z)$ for all analytic functions φ that satisfy this differential subordination. A dominant $\hat{q}$ is called the best dominant, if $q(z) \prec \hat{q}(z)$ for all the dominants q. For the general theory of the first-order differential subordination and its applications, we refer the reader to Bulboacă (2005) and Miller & Mocanu (2000).

DIFFERENTIAL SUBORDINATION ASSOCIATED WITH $H(n, \lambda; z)$

To establish our main results we shall require the following lemma.

Lemma 1 (Miller & Mocanu, 1985 and Miller & Mocanu, 2000). Let the function $q(z)$ be univalent in U, and let the functions θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set

$$Q(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

(i) $Q(z)$ is a starlike function in U,

(ii) $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \ (z \in U)$.

If p is analytic in U and $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2.1).

Theorem 1. Let $\lambda \in R\backslash\{0\}$, $n \in Z$ and $f(z) \in A$. Suppose also that the function $q(z)$ univalent in U, with $q(0) = 1$ and $q(z) \neq 0$ $(z \in U)$, and satisfies each of the following inequalities:

$$Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0 \quad (z \in U) \tag{2.2}$$

and

$$Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{\lambda}q(z)\right) > 0 \quad (z \in U). \tag{2.3}$$

If

$$H(n, \lambda; z) \prec q(z) + \lambda\frac{zq'(z)}{q(z)}, \tag{2.4}$$

then

$$\frac{I^{n-1}f(z)}{I^n f(z)} \prec q(z)$$

and $q(z)$ is the best dominant of (2.4).

Proof. We choose

$$g(z) = \frac{I^{n-1}f(z)}{I^n f(z)}, \quad \theta(w) = w \text{ and } \varphi(w) = \frac{\lambda}{w}.$$

Then $\theta(w)$ and $\varphi(w)$ are analytic in the domain $C^* = C\backslash\{0\}$, which contains $q(U)$, $q(0) = 1$, and $\varphi(w) \neq 0$ when $w \in q(U)$. Next, we define the functions $Q(z)$ and $h(z)$ by

$$Q(z) = zq'(z)\varphi(q(z)) = \lambda\frac{zq'(z)}{q(z)} \tag{2.5}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \lambda\frac{zq'(z)}{q(z)}. \tag{2.6}$$

It follows from (2.2) and (2.3) that $Q(z)$ is starlike in U and

$$Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \quad (z \in U).$$

We note also that the function $g(z)$ is analytic in U, with $g(0) = q(0) = 1$, since $0 \notin q(U)$. Therefore $g(U) \subset C^*$. Thus, the hypotheses of Lemma 1 are satisfied and we find that, if

$$\theta(g(z)) + zg'(z)\varphi(g(z)) = H(n, \lambda; z) \prec h(z), \tag{2.7}$$

then

$$\frac{I^{n-1}f(z)}{I^n f(z)} \prec q(z).$$

and $q(z)$ is the best dominants.

Remark 1. If $q(z) \in P$ and $\lambda > 0$, then we can omit the condition (2.3) in Theorem 1.

Remark 2. If $q(z) \in P$ and $\lambda < 0$, then we can omit the condition (2.2) in Theorem 1.

Theorem 2. For $\lambda > 0$, $n \in Z$ and $0 \leq \alpha < 1$, if $f(z) \in A$ and

$$H(n, \lambda; z) \prec \frac{(1-2\alpha)^2 z^2 + 2[(1-2\alpha) + \lambda(1-\alpha)]z + 1}{(1-z)[1+(1-2\alpha)z]}, \tag{2.8}$$

then the operator $I^n f(z)$ is a starlike function of order α in U, that is,

$$Re\frac{I^{n-1}f(z)}{I^n f(z)} > \alpha \qquad (z \in U).$$

Proof. For $0 \leq \alpha < 1$ and $z \in U$, we first put

$$q(z) = \frac{1+(1-2\alpha)z}{1-z} \in P$$

in Theorem 1. Then, since

$$Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = Re\left(\frac{1}{1-z} + \frac{1}{1+(1-2\alpha)z} - 1\right) > 0 \quad (z \in U),$$

the proof of Theorem 2 is completed.

Remark 3. Letting $\lambda \to 0^+$ in (2.8) , we have

$$\frac{I^{n-1}f(z)}{I^n f(z)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

which implies that the operator $I^n f(z)$ is a starlike function of order α in U, that is, $Re\left\{\frac{I^{n-1}f(z)}{I^n f(z)}\right\} > \alpha \quad (0 \le \alpha < 1,\ z \in U)$.

By taking $\alpha = 0$ in Theorem 2, we obtain the following result.

Corollary 1. For $\lambda > 0$ and $n \in Z$, if $f(z) \in A$ and

$$H(n,\ \lambda;\ z) \prec \frac{z^2+2(1+\lambda)z+1}{1-z^2}, \tag{2.9}$$

then the operator $I^n f(z)$ is a starlike function in U, that is,

$$Re\left(\frac{I^{n-1}f(z)}{I^n f(z)}\right) > 0 \ \ (z \in U).$$

Theorem 3. Let $\lambda < 0$, $0 \le \alpha < 1$, $n \in Z$, such that

$$A(\lambda,\ \alpha;\ z)Re(z) + B(\lambda,\ \alpha;\ z) > 0 \qquad (z \in U),$$

where

$$A(\lambda,\ \alpha;\ z) = -\lambda(1-2\alpha)|1-z|^2 + [\lambda + 2(1-\alpha)]|1+(1-2\alpha)z|^2, \tag{2.10}$$

and

$$B(\lambda,\ \alpha;\ z) = |1-z|^2[(1+\lambda-2\alpha)|1+(1-2\alpha)z|^2 - \lambda] - [\lambda+2(1-\alpha)]|1+(1-2\alpha)z|^2. \tag{2.11}$$

If $f(z) \in A$ and $H(n,\ \lambda;\ z)$ satisfies the subordination (2.8), then the operator $I^n f(z)$ is a starlike function of order α in U, that is,

$$Re\left(\frac{I^{n-1}f(z)}{I^n f(z)}\right) > \alpha \quad (z \in U;\ 0 \le \alpha < 1).$$

Proof. For $0 \le \alpha < 1$, $\lambda < 0$ and $z \in U$, we first set $q(z) = \frac{1+(1-2\alpha)z}{1-z}$ in Theorem 1, to obtain,

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{1}{\lambda}q(z)=\frac{1+[\lambda+(1-2\alpha)]z}{\lambda(1-z)}+\frac{1}{1+(1-2\alpha)z}.$$

Then, after some calculations, we observe that

$$Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{1}{\lambda}q(z)\right)=-\frac{A(\lambda,\alpha;z)Re(z)+B(\lambda,\alpha;z)}{\lambda|1-z|^2|1+(1-2\alpha)z|^2}>0 \quad (z\in U),$$

which completes the proof of Theorem 3.

By taking $\alpha=0$ in Theorem 3, we obtain the following result.

Corollary 2. For $\lambda<0$, such that

$$(\lambda+1)(1+|z|^2)+2Re(z)<0 \quad (z\in U). \tag{2.12}$$

If $f(z)\in A$ and the function $H(n,\lambda;z)$ satisfies the subordination relation (2.9) , then the operator $I^n f(z)$ is starlike in U .

Proof. For $\lambda<0$ and $z\in U$, we first set $q(z)=\frac{1+z}{1-z}$, in Theorem 1, to obtain,

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{1}{\lambda}q(z)=\frac{1+(\lambda+1)z}{\lambda(1-z)}+\frac{1}{1+z}.$$

Then, after some calculations, we observe that

$$Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{1}{\lambda}q(z)\right)=\frac{(1-|z|^2)[(\lambda+1)(1+|z|^2)+2Re(z)]}{\lambda|1-z|^2|1+z|^2}>0 \quad (z\in U),$$

which completes the proof of Corollary 2.

Finally, by taking $\alpha=0$ and $\lambda\leq-2$ in Theorem 3, we obtain the following sufficient condition for starlikeness.

Corollary 3. For $\lambda\leq-2$ and $z\in U$, if the function $H(n,\lambda;z)$ satisfies the subordination relation (2.9), then the operator $I^n f(z)$ is starlike in U.

Proof. Since $Re(z)\leq|z|<1$, $z\in U$, then from (2.12) for $\lambda\leq-2$, we have

$$(\lambda+1)(1+|z|^2)+2Re(z)<0 \quad (z\in U),$$

which proves Corollary 3.

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بعض تطبيقات التبعية التفاضلية على صنف خاص من الدوال التحليلية المعرفة بمؤثر تكاملي

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خلاصة

نقدم في هذا البحث دالة ذات وسيطين تشتمل على مؤثر فيلت الضارب. كما نحصل أيضاً على خواص التبعية والشروط الكافية لتكون هذه الدوال نجمية.

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